New sums of three cubes

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It is a long standing problem whether every integer $n \not\equiv 4, 5 \pmod 9$ may be written as a sum of three integral cubes. According to the web page `http://cr.yp.to/threecubes.html` of Daniel Bernstein, the first computational attacks were carried out as early as in 1955.

Nevertheless, for example for $n = 3$, there is still no solution known different from the obvious $(1, 1, 1)$, $(4, 4, -5)$, $(4, -5, 4)$, and $(-5, 4, 4)$. For $n = 30$, the first solution was found by N. Elkies and his coworkers in 2000 [El]. Note that, in 1992, D. R. Heath-Brown [HB] had made some prevision on the density of the solutions for $n = 30$ without knowing any solution, explicitly.

We implemented a version of Elkies' method described in [El] in `C` and `C++`. We took care that only parts of the code which are not time critical were written in `C++` using the multi precision floats of `GMP`.

The other parts were written in plain `C`. It turned out that, for most of the computations, 128-bit fixed-point arithmetic was sufficiently precise. We realized 128-bit fixed-point numbers as arrays consisting of two `long ints`.

We searched systematically for solutions of $x^3 + y^3 + z^3 = n$ where $n < 1000$ is neither a cube nor twice a cube and $|x|, |y|, |z| < 10^{14}$. For that, the curve $y = \sqrt[3]{1 - x^3}$, $x = [0, 1/\sqrt[3]{2}]$, was covered by small parallelograms, the so-called flagstones. The length of the flagstones was chosen dynamically. It was around $8.4 \cdot 10^{-12}$ near $x = 0$ and around $6.6 \cdot 10^{-14}$ near $x = 1/\sqrt[3]{2}$. The area of the flagstones was essentially constant at a value near $1.7 \cdot 10^{-40}$.

The whole search took around ten months of CPU time. Only 14% of that running time was spent on lattice reduction. The point is that the calculations were carried out highly efficiently in 128-bit fixed-point numbers. The lion's share of the time was spent on searching for small lattice points, i.e. on our implementation of the algorithm of Fincke-Pohst.

In comparison with the list, dating back to 2001 and published on `http://cr.yp.to/threecubes.html`, 3520 new solutions were found. Among them,

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there are solutions for $n = 52$, 156, 318, 366, 399, 420, 564, 641, 758, 789, 894, 948, and 996. For each of these numbers, no solution was known in 2001.

For example, our computations show

$$\begin{aligned} 52 &= 60\,702\,901\,317^3 + 23\,961\,292\,454^3 - 61\,922\,712\,865^3 \\ &= 1\,232\,911\,859\,663^3 + 343\,101\,441\,461^3 - 1\,241\,705\,896\,626^3. \end{aligned}$$

For 25 values of n, for which exactly one solution was known in 2001, we found a second one. Among those, there is $n = 30$. The second solution for $n = 30$ looks like this,

$$30 = 3\,982\,933\,876\,681^3 - 636\,600\,549\,515^3 - 3\,977\,505\,554\,546^3.$$

A second and a third solution for $n = 75$ are as follows,

$$\begin{aligned} 75 &= 2\,576\,191\,140\,760^3 + 1\,217\,343\,443\,218^3 - 2\,663\,786\,047\,493^3 \\ &= 59\,897\,299\,698\,355^3 - 47\,258\,398\,396\,091^3 - 47\,819\,328\,945\,509^3. \end{aligned}$$

A complete list of all 11 484 solutions we know for $n < 1000$, n being neither a cube nor twice a cube, is available from the second author's web page `http://www.uni-math.gwdg.de/jahnel` as the file `threecubes_20070419.txt`.

Unfortunately, we do not know of a solution for $n = 33$ or $n = 42$.

References

[El] Elkies, N. D.: *Rational points near curves and small nonzero* $|x^3 - y^2|$ *via lattice reduction,* in: Algorithmic number theory (Leiden 2000), Lecture Notes in Computer Science 1838, Springer, Berlin 2000, 33–63

[HB] Heath-Brown, D. R.: *The density of zeros of forms for which weak approximation fails,* Math. Comp. **59** (1992), 613–623