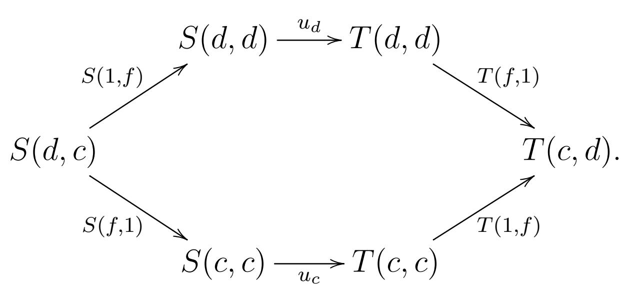
Dinatural Transformations

Dinatural transformations can be viewed as an attempt to generalize the notion of a natural transformation to functors that needn't be of the same variance. Here is the general concept:

Definition. Given functors $S, T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, a **dinatural transformation** $u : S \stackrel{..}{\to} T$ is a family of morphisms in $\mathcal{D}$,

$$u_c : S(c,c) \to T(c,c), \qquad c \in \mathcal{C},$$

such that for every $f : c \to d$ in $\mathcal{C}$ we have a commutative hexagon

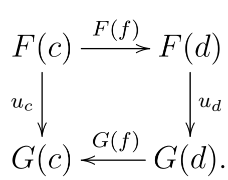


Clearly, the restriction to the diagonal of a natural transformation $u : S \Rightarrow T$ is dinatural.

In line with the introduction, one considers functors $\mathcal{C} \to \mathcal{D}$ and $\mathcal{C}^{op} \to \mathcal{D}$ and considers dinatural transformations of the functors obtained by precomposing with projections from $\mathcal{C}^{op} \times \mathcal{C}$. For example, given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $F \Rightarrow G$ is just a dinatural transformation $F \circ \mathrm{pr}_2 \stackrel{..}{\to} G \circ \mathrm{pr}_2$. Similarly for contravariant functors, using pr_1. One also obtains a notion of "transformation" from a covariant functor F to a contravariant functor G: it is a dinatural transformation $F \circ \mathrm{pr}_2 \stackrel{..}{\to} G \circ \mathrm{pr}_1$, which means we have

$$u_c : F(c) \to G(c), \qquad c \in \mathcal{C},$$

so that for every $f : c \to d$ in $\mathcal{C}$ we have a commutative diagram



Wedges and Coends

We will be interested in dinatural transformations from or to a constant functor $e \in \mathcal{D}$, also called **wedges** (or extranatural transformations).

Explicitly, $u : S \stackrel{..}{\to} e$ consists of arrows

$$u_c : S(c,c) \to e, \qquad c \in \mathcal{C}$$

so that for every $f : c \to d$ in $\mathcal{C}$ we have

$$\begin{array}{ccc} S(d,c) & \xrightarrow{S(1,f)} & S(d,d) \\ {\scriptstyle S(f,1)}\big\downarrow & & \big\downarrow{\scriptstyle u_d} \\ S(c,c) & \xrightarrow[u_c]{} & e. \end{array}$$

Definition. A **coend** of a functor $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is an object, written $\int^c S(c,c) \in \mathcal{D}$, together with an initial wedge $u : S \stackrel{..}{\to} \int^c S(c,c)$.

That is, u is a wedge $u_c : S(c,c) \to \int^c S(c,c)$ so that for any other wedge $w_c : S(c,c) \to f$ there exists a unique arrow $\phi : \int^c S(c,c) \to f$ in $\mathcal{D}$ with $\phi \circ u_c = w_c$ for all $c \in \mathcal{C}$.

In other words, a coend of $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is just a coequalizer

$$\begin{array}{ccccc} S(d,c) & \xrightarrow{S(1,f)} & S(d,d) & & \\ {\scriptstyle \mathrm{in}_f}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{in}_d} & & \\ \coprod_{f:c\to d \text{ in } \mathcal{C}} S(d,c) & \rightrightarrows & \coprod_{c\in\mathcal{C}} S(c,c) & \longrightarrow & \int^c S(c,c). \\ {\scriptstyle \mathrm{in}_f}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{in}_c} & & \\ S(d,c) & \xrightarrow[S(f,1)]{} & S(c,c) & & \end{array}$$

In particular, in a co-complete category all coends exist. Conversely, if $F : \mathcal{I} \to \mathcal{C}$ is a diagram, then a wedge from $F\mathrm{pr}_2$ is just a co-cone, so

$$F(i) = F\mathrm{pr}_2(i,i) \to \int^i F\mathrm{pr}_2(i,i), \qquad i \in \mathcal{I},$$

is a colimit of F; thus the existence of all coends implies that $\mathcal{C}$ is co-complete. The coequalizer description of the coend shows also that a co-continuous functor preserves all coends.

Let $F : \mathcal{C}^{op} \to \mathcal{M}$ and $G : \mathcal{C} \to \mathcal{M}$ be functors into a monoidal category $(\mathcal{M}, \otimes)$. Then the coend $\int^c F(c) \otimes G(c)$ of $\otimes \circ (F \times G)$ is called the **tensor product** of F and G and written $F \otimes G$.

Example. $(\mathbf{Top}, \times)$ is monoidal. Let $X \in \mathbf{S} = \mathbf{Cat}(\Delta^{op}, \mathbf{Set})$ be a simplicial set and consider also the functor $\Delta \to \mathbf{Top}$, $\mathbf{n} \mapsto |\Delta^n|$. A coend of the functor

$$\Delta^{op} \times \Delta \to \mathbf{Top}, (\mathbf{n}, \mathbf{m}) \mapsto X_n \times |\Delta^m|$$

is called the **geometric realization** of X:

$$|X| = X \times |\Delta^\bullet| = \int^{\mathbf{n}} X_n \times |\Delta^n|.$$

Example. Let $\mathcal{M}$ be a simplicial model category. In particular, $\mathcal{M}$ is tensored over $\mathbf{S}$, given by a functor $\otimes : \mathcal{M} \times \mathbf{S} \to \mathcal{M}$. If $F : \mathcal{I} \to \mathcal{M}$ is a diagram, then the **homotopy colimit** is by definition

$$\mathrm{hocolim}_{\mathcal{I}} F = \int^i F(i) \otimes B(i \downarrow \mathcal{I})^{op} \in \mathcal{M}$$

Here $B(- \downarrow \mathcal{I})^{op}$ is viewed as the functor $\mathcal{I}^{op} \to \mathbf{S}$ which associates to each $i \in \mathcal{I}$ the classifying space of the comma category $(i \downarrow \mathcal{I})^{op}$. Thus

$$B(i \downarrow \mathcal{I})^{op}_n = \{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n \leftarrow i \text{ in } \mathcal{I}\}.$$

Ends

A dinatural transformation $u : e \stackrel{..}{\to} S$, also called a wedge, consists of arrows

$$u_c : e \to S(c, c), \qquad c \in \mathcal{C},$$

so that for every $f : c \to d$ we have a commutative diagram

$$\begin{array}{ccc} e & \xrightarrow{u_c} & S(c,c) \\ {\scriptstyle u_d}\downarrow & & \downarrow{\scriptstyle S(1,f)} \\ S(d,d) & \xrightarrow[S(f,1)]{} & S(c,d). \end{array}$$

Definition. An end of a functor $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is an object $\int_c S(c,c) \in \mathcal{D}$ together with a final wedge $u : \int_c S(c,c) \dot{\dot{\to}} S$.

That is, u is a wedge $u_c : \int_c S(c,c) \to S(c,c)$ so that for any other wedge $w_c : f \to S(c,c)$ there exists a unique arrow $\phi : f \to \int_c S(c,c)$ in $\mathcal{D}$ with $u_c \circ \phi = w_c$ for all $c \in \mathcal{C}$.

The homotopy limit is an example of an end.

Properties of Coends

We will say that a functor $U : \mathcal{D} \to \mathcal{E}$ **creates coends**, if for every functor $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ and final wedge $u : US \dot{\dot{\to}} e$

1. there exists a unique wedge $v : S \dot{\dot{\to}} d$ with $Uv = u$ to some $d \in \mathcal{D}$
2. this unique wedge is already final.

The next proposition asserts that a "parameter-dependent" coend $\int^c S(c,c,p)$ of a functor $S : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{P} \to \mathcal{D}$ is functorial in p.

Proposition. *Suppose $S(-,-,p)$ has a coend for each $p \in \mathcal{P}$. Then the coend of $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}^{\mathcal{P}}$ may be computed pointwise.*

More precisely, the forgetful functor $\mathcal{D}^{\mathcal{P}} \to \mathcal{D}^{Ob(\mathcal{P})}$ creates all coends, viewing $Ob(\mathcal{P})$ as a discrete category. In particular, the coend may be viewed as a functor

$$\int : \mathbf{Fun}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) \to \mathcal{D}.$$

Proof. Suppose we are given wedges $u_c(p) : S(c,c,p) \to e(p)$ for each $p \in \mathcal{P}$. We wish to find a unique extension to a wedge $u_c : S(c,c,-) \dot{\dot{\to}} e$ in $\mathcal{D}^{\mathcal{P}}$ for an extension e of the object function $e(p)$. Necessarily, for $f : p \to q$ in $\mathcal{P}$,

$$\begin{array}{ccc} S(c,c,p) & \xrightarrow{u_c(p)} & e(p) \\ {\scriptstyle S(1_c,1_c,f)}\downarrow & & \downarrow{\scriptstyle e(f)} \\ S(c,c,q) & \xrightarrow{u_c(q)} & e(q), \end{array}$$

so if u_c is to be a wedge in $\mathcal{D}^{\mathcal{P}}$, there is just one way to define e on morphisms, namely as the map induced on the coend $e(p)$ by the wedge $u_c(q) \circ S(1_c, 1_c, f)$.

It is now easy to check that the forgetful functor creates coends. The last assertion follows by taking $\mathcal{P} = [1]$, viewing a natural transformation as a functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}^{[1]}$. □

Next is a Fubini-type property of coends.

Proposition. *Let* $S : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{E}$ *be a functor and suppose each* $S(-, -, p, q)$ *has a coend* $\int^c S(c, c, p, q)$*. Then*

$$\int^p \int^c S(c, c, p, p) = \int^{(c,p)} S(c, c, p, p),$$

meaning that S *has a coend iff* $\int^c S(c, c, -, -)$ *has a coend and that they coincide. If in addition each* $S(c, d, -, -)$ *has a coend,*

$$\int^p \int^c S(c, c, p, p) = \int^c \int^p S(c, c, p, p).$$

Proof. A wedge $S(c, c, p, p) \stackrel{..}{\to} e$ is dinatural in c (and in p) and thus yields a wedge $\int^c S(c, c, p, p) \stackrel{..}{\to} e$. Conversely, a wedge $\int^c S(c, c, p, p) \stackrel{..}{\to} e$ (dinatural in p) may be precomposed with $S(c, c, p, p) \to \int^c S(c, c, p, p)$ to make a wedge on S. The proof is an easy diagram chase, where one uses to existence of the coend of $S(-, -, p, q)$ for *all* p, q to insert terms into the diagram so one is able to use the functoriality of S. □