Euler Class

Let $E \to B$ be an oriented rank $n$ vector bundle, where $E, B$ are CW-complexes.

1.1 Definition by Obstruction Theory

The Euler class $e \in H^n(B; \mathbb{Z})$ of $E \to B$ is the primary obstruction for finding a section of the fiber bundle

$$\pi : E_0 := E \setminus \{\text{zero section}\} \to B.$$ 

A nowhere vanishing section $s^{(n-1)} : B^{(n-1)} \to E$ on the $(n-1)$-skeleton always exists, proceeding by induction: $s^{(0)}$ is obtained by picking points in the fibers above the 0-skeleton. Given $s^{(k)}$ for $k < n-1$, consider the diagram of attaching maps

$$\begin{array}{cccc}
\coprod_{i \in I_n} S^k & \xrightarrow{\coprod \phi_i} & B^{(k)} & \xrightarrow{s^{(k)}} E_0 \\
\downarrow & & \downarrow & \downarrow \pi \\
\coprod_{i \in I_n} D^k & \xrightarrow{\coprod \Phi_i} & B^{(k+1)} & \to B.
\end{array}$$

Since $\pi \circ s^{(k)} \circ \phi_i : S^k \to B$ extends over $D^k$, it is nullhomotopic. Because $\pi$ is a fibration, this homotopy may be lifted to a homotopy from $s^{(k)} \circ \phi_i : S^k \to E_0$ to a map into some fiber $\psi_i : S^k \to \pi^{-1}(b)$. Now $\pi^{-1}(b) \approx \mathbb{R}^n \setminus \{0\}$ is $(n-1)$-connected and $k < n-1$, so the map $\psi_i$ is nullhomotopic, as is $s^{(k)} \circ \phi_i$ then, which thus extends to $D^k$. These extensions of $s^{(k)} \circ \phi_i$ along with $s^{(k)}$ give the required map on the pushout $B^{(k+1)}$. In case $k = n-1$ we extract for every $(n-1)$-cell $i \in I_{n-1}$ the degree of the map $\psi_i$ (using the orientation on each fiber) to get an association

$$I_n \to \mathbb{Z}$$

which is just a cellular $n$-cochain $e$ on $B$.

1.2 Definition by Thom Classes

Pick orientation-preserving bundle isomorphisms $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$. We obtain generators

$$\xi_\alpha \in H^n(E|_{U_\alpha}, (E|_{U_\alpha})_0; \mathbb{Z})$$

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which are independent of the choice of isomorphism. Similarly, we obtain generators

$$\xi_b \in H^n(E_b, E_b \setminus \{0\}; \mathbb{Z})$$

in each fiber, $b \in B$, and the restriction of $\xi_\alpha$ to any of its points $b \in U_\alpha$ is $\xi_b$. Using the Mayer-Vietoris sequence, one may piece together the so-called orientation class or Thom class

$$\xi \in H^n(E, E_0; \mathbb{Z})$$

which restricts to the elements $\xi_b$ and $\xi_\alpha$. This class induces the Thom isomorphism:

$$H^*(B; \mathbb{Z}) \xrightarrow{\cong} H^{*+n}(E, E_0; \mathbb{Z}), b \mapsto \pi^*(b) \cup \xi$$

The Euler class is the pullback of the Thom class along the zero section $s : (B, \emptyset) \to (E, E_0): e = s^*\xi$. 

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