Zeta Function of Graphs

Alireza Sarveniazi/Stefan Wiedmann

Mathematics Institute Universitity of Goettingen

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Basic definitions

First we start with some basic definitions and facts about graphs.

Let $X = (V, E)$ be a graph with VX as the set of vertices and EX as the set edges.
Assume that $|VX| = r_0 < \infty$, $|EX| = 2r_1 < \infty$ and that the graph X is directed in the sense that if $e \in EX$ then the inverse edge also is in EX.

Definition (Adjacency Matrix)

The adjacency matrix of the graph X is the matrix A indexed by the pairs of vertices $x, y \in V$, such that $A = (a_{xy})$, where

$$a_{xy} = \text{ number of edges joining } x \text{ to } y.$$

Regular and simple graphs

A graph is called simple if there is at most one edge joining (adjacent) vertices and if there are no loops; hence, X is simple if and only if $a_{xy} \in \{0, 1\}$ for all $x, y \in V$ and $a_{xx} = 0$ for all $x \in V$.

If every vertex x of X possesses exactly k edges departing (or arriving) from x then X is called k-regular.

In this talk all graphs will be simple and regular.

Spectrum of a graph

For a finite graph X on r_0 vertices, the adjacency matrix A is a symmetric matrix, hence it has r_0 real eigenvalues. Counting multiplicities we may list the eigenvalues in decreasing order:

$$\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{r_0 - 1}.$$

The spectrum of X is the set of eigenvalues of A. If X is k-regular we have:

- $\lambda_0 = k$;
- λ_0 has multiplicity $1 \iff X$ is connected;
- $-k \leq \lambda_i \leq k$ for $i = 0, 1, \ldots, r_0 - 1$.

Set $\lambda(X) := \max_{\lambda_i \neq \pm k} \{|\lambda_i|\}$.

Ramanujan graphs

For families of k-regular graphs with increasing number of vertices we have the following result:

Theorem (Alon - Boppana (1986))

Let $\{X_m\}_{m\geq 1}$ be a family of connected, k-regular, finite graphs, with $|V_m| \to \infty$ as $m \to \infty$. Then

$$\liminf_{m\to\infty} \lambda(X_m) \geq 2\sqrt{k-1}.$$

This bound gives us a motivation for the following definition:

Definition

A finite, connected, k-regular graph X is a Ramanujan graph if, for every nontrivial eigenvalue λ of X, one has $|\lambda| \leq 2\sqrt{k-1}$.

Some operators

For any field K we can associate to our graph X the K-vectorspaces:

- $C_0(X)$: K-vectorspace with base VX ($\dim = r_0$).
- $C_1(X)$: K-vectorspace with base EX ($\dim = 2 * r_1$).

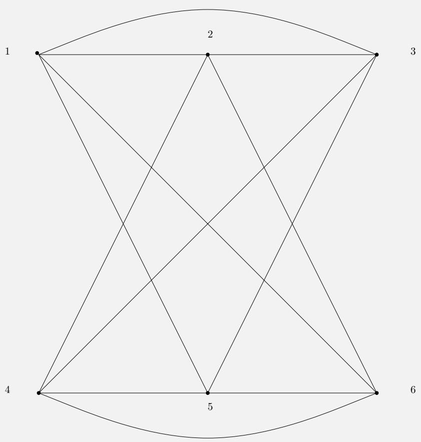
Now we can define the following operators:
Vertex-Adjacency:

$$A : C_0(X) \longrightarrow C_0(X)$$
$$x \longmapsto \sum_{\partial_0(e)=x} \partial_1(e)$$

Edge-Adjacency:

$$T : C_1(X) \longrightarrow C_1(X)$$
$$e \longmapsto \sum_{(e,e_1)_{red}} e_1$$

4-regular finite upper half plane graph $X_3(1,-1)$ with 6 Verticies.



Adjacency matrix and spectrum of $X_3(1,-1)$

Adjacency matrix:

```
0 1 1 0 1 1
1 0 1 1 0 1
1 1 0 1 1 0
0 1 1 0 1 1
1 0 1 1 0 1
1 1 0 1 1 0
```

Spectrum:

```
Eig := [-2,-2,0,0,0,4];
```

We have $2 \leq 2\sqrt{3} \Longrightarrow$ Ramanujan graph.

Matrix T of $X_3(1, -1)$

```
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
```


Magma code I

```
 //Construction of edge adjacency out of vertex adjacency

//First: Construct a list of the orientated egdges

function ad2T(A)
    n := NumberOfRows(A);
    L :=[];
    for i:=1 to n do
for j:= 1 to n do
    if i ne j then
if A[i,j] ne 0 then
    Append(~L,[i,j]);
    L1 :=[IntegerRing()|];
    for k:=1 to n do
if k eq i then
    continue;
end if;
if A[j,k] ne 0 then
    Append(~L1,k);
end if;
    end for;
    Append(~L,L1);
end if;
    end if;
end for;
    end for;
    return L;
end function;
```


Magma code II

```
//Second: Construct the edge adjacency matrix out of the above list.

function L2T(L)
    n := Round(#L/2);
    print n;
    A := ZeroMatrix(IntegerRing(),n,n);
    for t:=1 to n do
for s:=1 to #L[2*t] do
    A[Round((Index(L,[L[2*t-1][2],L[2*t][s]])+1)/2),t]:=1;
end for;
    end for;
    return A;
end function;
```


Explicit construction of families of Ramanujan graphs

All known explicit constructions are using arithmetical number theory. These constructions exist only if k is of the form $k = p^n + 1$, where p is a prime number and $n \geq 1$.

Open question

Do exist an explicit construction of a family of Ramanujan graphs for all $k \geq 3$?
The first interesting case will be a family of 7-regular Ramanujan graphs.

We are considering here the following families of Ramanujan graphs:

- LPS graphs.
- Finite upper half plane graphs.

Cayley graphs

Let G be a group (finite or infinte) and let S be a nonempty, finite subset of G. We assume that S is symmetric; that is, $S = S^{-1}$.

Definition

The Cayley graph $\mathcal{C}(G, S)$ is the graph with vertex set $V = G$ and edge set

$$E := \{(x, y) \mid x, y \in G;\ \exists s \in S \text{ s.t. } y = xs\}$$

We define $k := |S|$.

Properties of Cayley graphs

Let $\mathcal{C}(G, S)$ be a Caley graph.

- $\mathcal{C}(G, S)$ is a simple, k-regular, vertex-transitive graph.
- $\mathcal{C}(G, S)$ has no loop if and only if $1 \notin S$.
- $\mathcal{C}(G, S)$ is connected if and only if S generates G.
- If there exsts a homomorphism ψ from G to the multiplicative group $\{-1, 1\}$, such that $\psi(S) = -1$, then $\mathcal{C}(G, S)$ is bipartite. The converse holds provided $\mathcal{C}(G, S)$ is connected.

LPS family of Ramanujan graphs I

Let $p \equiv 1 \bmod 4$ be a prime number. Jacobi's theorem implies that the set of integer solutions

$$S := \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = p \text{ and } x_0 \geq 0, x_0 \text{ is odd and } x_i \text{ is even for } i = 1, 2, 3\}$$

has exactly $p + 1$ elements.
Let $q \equiv 1 \bmod 4$ be a another prime number. We associate to S a set of generators in the groups $\mathrm{PSL}_2(\mathbb{F}_q)$ and $\mathrm{PGL}_2(\mathbb{F}_q)$ respectively. Let $i \in \mathbb{F}_q$ be an element s.t. $i^2 = -1$.

LPS family of Ramanujan graphs II

- $\left(\frac{p}{q}\right) = 1$ i.e, the equation $x^2 = p$ has a solution δ in $\mathbb{F}_q$.

$$\tilde{S} := \left\{ \frac{1}{\delta} \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \text{ s.t. } (x_0, x_1, x_2, x_3) \in S \right\} \subseteq \mathrm{PSL}_2(\mathbb{F}_q)$$

- $\left(\frac{p}{q}\right) = -1$ i.e, the equation $x^2 = p$ has no solution in $\mathbb{F}_q$.

$$\tilde{S} := \left\{ \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \text{ s.t. } (x_0, x_1, x_2, x_3) \in S \right\} \subseteq \mathrm{PGL}_2(\mathbb{F}_q)$$

The graph $X^{(p,q)}$

We define the Cayley graph $X^{(p,q)}$ as follows:

$$X^{(p,q)} := \begin{cases} \mathcal{C}(\mathrm{PSL}_2(\mathbb{F}_q), \tilde{S}) & \text{if } \left(\frac{p}{q}\right) = 1, \\ \mathcal{C}(\mathrm{PGL}_2(\mathbb{F}_q), \tilde{S}) & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

The latter will be bipartit.

The Theorem of Lubotzky-Philips-Sarnak

For all primes p, q as above we have the following theorem:

Theorem (Lubotzky-Philips-Sarnak (1988))

$X^{p,q}$ is a $(p+1)$-regular Ramanujan graph.

Sketch of the proof I

- The Bruhat-Tits tree $\mathcal{T}_p$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ is a universal covering of the $(p+1)$ regular graph $X^{p,q}$ and

  $$X^{p,q} \cong \Gamma(q)\backslash\mathcal{T}_p$$

  where

  $$\Gamma(q) := \ker(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \longrightarrow \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]/q\mathbb{Z}[\frac{1}{p}]))$$

  is a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Q}_p)$.

Sketch of the proof II

- Interpretation of the adjacency matrix of $X^{p,q}$ as a Hecke operator defined over cusp forms of weight 2 of the congruence subgroup $\Gamma(q)$.
- Now the proof will be complete from Deligne's theorem about Ramanujan-Petersson conjecture:
  All eigenvalues of the Hecke operator defined over the cusp forms of weight k of the congruence subgroup $\Gamma(q)$ satisfies: $|\lambda| \leq 2p^{\frac{k-1}{2}}$, set $k = 2$.

In fact the Ramanujan-Petersson conjecture is a result of Weil's conjecture which was proved by Deligne in 1973. Deligne's proof uses a pure algebraic geometric method, up to date there is no other proof.

The family of finite upper half plane graphs (Terras family)

Let $\mathbb{F}_q$ be a finite field of order q and $\delta \in \mathbb{F}_q$ a nonsquare.
The finite upper half plane $\mathbb{H}_q$ is by definition:

$$\mathbb{H}_q := \{z = x + \sqrt{\delta}y \mid x, y \in \mathbb{F}_q,\ y \neq 0\}$$

Let be $a \in \mathbb{F}_q$ with $a \neq 0, 4\delta$. Define a graph $X_q(a, \delta)$ in the following way:
The set of vertices V are the points of the upper half plane $\mathbb{H}_q$.
Two points $z_1, z_2 \in \mathbb{H}_q$ are adjacent if $d(z_1, z_2) = a$ where

$$d(z_1, z_2) := \frac{N(z_1 - z_2)}{\Im(z_1)\Im(z_2)},\ N(x+\sqrt{\delta}y) := x^2 - \delta y^2,\ \Im(x+\sqrt{\delta}y) := y.$$

The upper half plane graph $X_q(a, \delta)$ is Ramanujan

The graph $X_q(a, \delta)$ is $(q + 1)$-regular. In fact it is a Caley graph:

$$X_q(a, \delta) = \mathcal{C}(\mathrm{Aff}_2(q), S_q(a, \delta))$$

where

$$\mathrm{Aff}_2(q) := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{F}_q\,, y \neq 0 \right\}$$

and

$$S_q(a, \delta) := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in Aff_2(q)\ :\ x^2 = ay + \delta(y-1)^2 \right\}$$

For $a \neq 0, 4\delta$ the equation $x^2 = ay + \delta(y-1)^2$ has exactly $q + 1$ solution in $\mathbb{F}_q$. A. Terras conjectured that they are Ramanujan graphs, and this was proved by N. Katz:
Estimates for Soto-Andrade sums, J. reine angew. Math. 438. (1993).

Our main goal I

Theorem (A. Sarveniazi)

Let $\mathcal{F} = \{X_p\}_{p\in\mathcal{P}}$ be a certain family Ramanujan where $\mathcal{P}$ is the set of all prime numbers. For a complex number s with $\Re(s) > 2$, the Riemann zeta function $\zeta(s)$ satisfies:

$$\zeta(s) = \prod_{p\in\mathcal{P}} (1 + p^{1-s})^{\frac{2}{p-1}} \exp\left(\sum_{p\in\mathcal{P}} \frac{1}{-\chi_p} \sum_{m=1}^{\infty} \frac{B^{\mathcal{F}}_{p,m}(s)}{m} p^{\frac{-ms}{2}} \right) \qquad (1)$$

where

$$B^{\mathcal{F}}_{p,m}(s) := C_m(X_p) - \frac{tr(A^m_{X_p})}{(1 + p^{1-s})^m}\ .$$

$C_m(X_p)$ is the number of closed and tail-less paths without backtracking in X_p and $\chi_p := r_{0,p} - r_{1,p}$.

Our main goal II

- Find the most efficient way to calculate Riemann zeta function and its zero's from the formula (1) using a certain family of Ramanujan graphs.
  - Find the most efficient algorithm to calculate $C_n(X_p)$ for possibly large numbers n and prime numbers p.
  - Find the best family of Ramanujan graphs with the most efficient algorithm in order to calculate $C_n(X_p)$.
  - Determine the distribution of C_n's for each family.
  - Compare the growth of C_n's between different families.

Paths and cycles I

- A path form x to y: Tuple of edges $\gamma = (e_1, \dots, e_n)$ s.t. $\partial_0(e_1) = x$, $\partial_1(e_n) = y$ and $\partial_1(e_{i-1}) = \partial_0(e_i)$ for $1 < i \leq n$. Length of a path $l(\gamma) = n$.
- Backtrack: $\overline{e_{i+1}} = e_i$.
- Tail: $\overline{e_n} = e_1$.
- Cycle: $\partial_0(e_1) = \partial_1(e_n)$.

Paths and cycles II

- $R_n^x :=$ $\#\{\text{Cycles around } x \text{ without backtrack (reduced) of length } n\}$.
- $C_n^x :=$ $\#\{\text{Cycles around } x \text{ without backtrack and tail of lenght } n\}$.
- Primitive cycle: Not of the shape $(e_1, \dots, e_k, e_1, \dots, e_k, \dots, e_1, \dots, e_k)$.
- Cycles are eqivalent iff: $(e_1, \dots, e_n) \sim (e_i, \dots, e_n, e_1, \dots, e_{i-1})$.
- $\mathcal{P}$: Eqivalenceclasses of primitive cycles.

Zeta function of graphs (Ihara 1966)

Definition (Zeta function)

$$Z(X,u) := \prod_{\gamma \in \mathcal{P}} \frac{1}{1 - u^{l(\gamma)}}$$

If we define $q := k - 1$ and $u := q^{-s}$:

$$Z(X,s) = \prod_{\gamma \in \mathcal{P}} \frac{1}{1 - q^{-sl(\gamma)}}.$$

Identities for the Zeta function

Lemma

$$Z(X,u) = \exp\left(\sum_{n=1}^{\infty} C_n \frac{u^n}{n}\right)$$

$$Z(X,u) = \frac{1}{\det(I_{r_0} - uT)}.$$

where $\mathrm{Tr}(T^n) = C_n$ *for* $n > 0$.

Corollary

The Zeta function is a rational function without zeros. The poles are the eigenvalues of T.

Calculation-Algorithm of C_n I

1. Calculate the adjacency matrix A
2. Calculate from the matrix A the edge adjacency matrix T
3. Calculate the eigenvalues of T using *matlab* or *octave*
4. Calculate the values of C_n as a sum of powers of these eigenvalues.

Matrix T associated to $X_3(1,-1)$

```
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
```


Cycles (around x) for $X_3(1,-1)$ - calculation using the trace of T^n

$C_n/6 = C_n^x$ for $n = 1, \ldots, 40$

```
0, 0, 8, 20, 40, 112, 336, 1188, 3344, 9440, 29656,
89524, 264888, 795984, 2391968, 7177028, 21529888,
64549312, 193692840, 581227860, 1743368264,
5229933104, 15690734320, 47071910884, 141214574640,
423643727520, 1270931460728, 3812803380980
11438400876760, 34315165468432, 102945569829696,
308836760219268, 926510045017664, 2779530195862400,
8338590914608456, 25015772682195604, 75047317963623528,
225141951917163504, 675425857623271568,
2026277582020942628
```


Calculation-Algorithm of C_n II

1. Calculate the adjacency matrix A
2. Calculate from the matrix A the matrix A_n defined as follows: For $x, y \in X$ we define

   $A_n(x,y) := \#$ Paths without backtracks from x to y of length.

   It is an exercise to proof:

$$
\begin{aligned}
A_1 &= A \\
A_2 &= A_1^2 - kE_{r_0} \\
A_{n+1} &= A_1 A_n - (k-1)A_{n-1} \text{ if } n > 2.
\end{aligned}
$$

Calculation of C_n at x using the matrices A_n

Let y be a vertex connected with x.

- $R_n^{xy} := \#\{\text{Red. paths from } x \text{ to } y \text{ of length } n\} = (A_n)_{xy}$.
- $K_n^y :=$
  $\#\{\text{Red. cycles around } y \text{ of length } n \text{ s.t. } e_1 \neq e_{yx} \text{ and } e_n \neq e_{xy}\}$.
- $L_n^{xy} := \#\{\text{ Red. paths form } x \text{ to } y \text{ of lenght } n \text{ s.t. } e_1 \neq e_{xy}\}$.

Calculation of C_n around x using the matrices A_n

We can proof:

1. $C_n^x = R_n^x - kK_{n-2}^y$.
2. $K_n^y = R_n^y - 2L_{n-1}^{xy} + K_{n-2}^x$.
3. $L_n^{xy} = R_n^{xy} - R_{n-1}^x + L_{n-2}^{xy}$.

Inital values:

1. $L_1^{xy} = 0,\ L_2^{xy} = R_2^{xy}$.
2. $K_1^y = 0,\ K_2^y = 0$.
3. $C_1^x = 0,\ C_2^x = 0$.

Works well with Terras family: $p = 3,\ n = 1, \dots, 100$ and for $p = 3, 5, 7, 11,\ n = 1, \dots, 10$.

Ihara-Bass formula

Theorem

Let be $\chi := r_0 - r_1$, $q := k - 1$ and let I_{r_0} be the identity matrix. Then:

$$Z(X, u) = \frac{(1 - u^2)^{\chi}}{\det(I_{r_0} - uA + u^2 q I_{r_0})}.$$

From the lemma above we conlcude:

Corollary

$$Z(X, u) = \frac{1}{\det(I_{2r_1} - uT)} = \frac{(1-u^2)^{\chi}}{\det((1+u^2 q)I_{r_0} - uA)}$$

Calculation-Algorithm for C_n using Ihara-Bass formula

1. Calculate the adjacency matrix A.
2. Calculate the eigenvalues of A using *Matlab* or *Octave*.
3. From Ihara Bass formula, we have:

$$\begin{aligned}\mathrm{Eig}(T) = {} & [1/2(\lambda + i\sqrt{4p - \lambda^2} \mid \lambda \in \mathrm{Eig}(A)] \\ & \cup [1/2(\lambda - i\sqrt{4p - \lambda^2} \mid \lambda \in \mathrm{Eig}(A)] \\ & \cup [\underbrace{1, \dots, 1}_{-\chi \text{ pieces}}] \\ & \cup [\underbrace{-1, \dots, -1}_{-\chi \text{ pieces}}].\end{aligned}$$

4. Calculate the values of C_n as a sum of powers of these eigenvalues.

Test of Ihara-Bass formula for $X_3(1, -1)$

```
Determinant(1-uT) =
729*u^24 - 3888*u^22 - 432*u^21 + 7938*u^20 + 2160*u^19
- 6912*u^18 - 4032*u^17 + 639*u^16 + 3008*u^15 + 2976*u^14
+ 96*u^13 - 1412*u^12 - 1248*u^11 - 384*u^10 + 320*u^9
+ 327*u^8 + 192*u^7 + 16*u^6 - 48*u^5 - 30*u^4 - 16*u^3 + 1
```

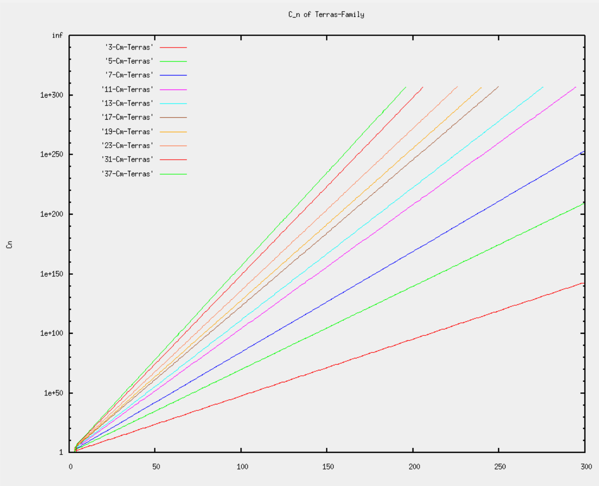
```
Determinant(1-u*A+3*u^2)*(1-u^2)^6 =
729*u^24 - 3888*u^22 - 432*u^21 + 7938*u^20 + 2160*u^19
- 6912*u^18 - 4032*u^17 + 639*u^16 + 3008*u^15 + 2976*u^14
+ 96*u^13 - 1412*u^12 - 1248*u^11 - 384*u^10 + 320*u^9 +
327*u^8 + 192*u^7 + 16*u^6 - 48*u^5 - 30*u^4 - 16*u^3 + 1
```


Factorisation of $\det(1 - uT)$ and of $\det(1 - uA + pu^2)$

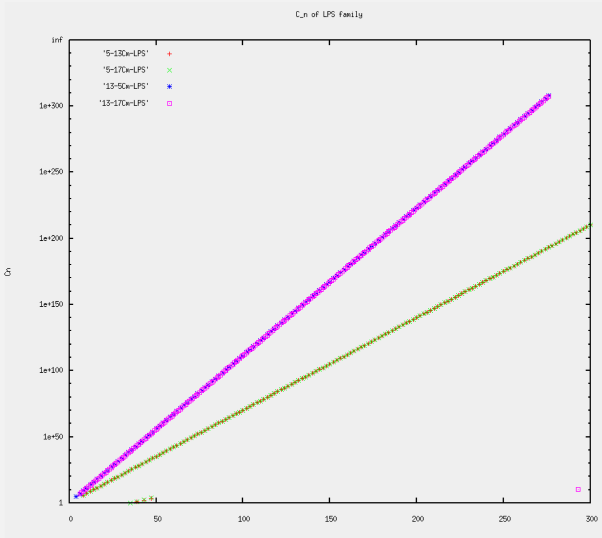
```
 > f:=Determinant(1 - uT);
> f;
729*u^24 - 3888*u^22 - 432*u^21 + 7938*u^20 + 2160*u^19 - 6912*u^18 - 4032*u^17
     + 639*u^16 + 3008*u^15 + 2976*u^14 + 96*u^13 - 1412*u^12 -
    1248*u^11 - 384*u^10 + 320*u^9 + 327*u^8 + 192*u^7 + 16*u^6 - 48*u^5 - 30*u
        ^4 - 16*u^3 + 1
> Factorisation(f);
[
    <u - 1, 7>,
    <u + 1, 6>,
    <3*u - 1, 1>,
    <3*u^2 + 1, 3>,
    <3*u^2 + 2*u + 1, 2>
]
```

```
 > f:=Determinant(1 - u*A+3 *u^2);
> f;
729*u^12 + 486*u^10 - 432*u^9 - 81*u^8 - 432*u^7 - 108*u^6 - 144*u^5 - 9*u^4 -
     16*u^3 + 6*u^2 + 1
> Factorisation(f);
[
    <u - 1, 1>,
    <3*u - 1, 1>,
    <3*u^2 + 1, 3>,
    <3*u^2 + 2*u + 1, 2>
]
```


Cn's distribution of Terras family



Cn's distribution of LPS family



Images of Terras family

Thanks to Tim Oliver Kaiser we present some images of the Terras family.