Saito's result

Theorem (K. Saito 1971): Let $(X, 0)$ be the germ of an isolated hypersurface singularity. The following conditions are equivalent:

- $(X, 0)$ is quasi-homogeneous.
- $\mu(X, 0) = \tau(X, 0)$.
- The Poincaré complex of $(X, 0)$ is exact.

We wanted to generalize this theorem to the case of isolated complete intersection singularities.

SINGULAR and Applications

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Poincaré complex

Let $(X_{l,k}, 0)$ be the germ of the unimodal space curve singularity $FT_{k,l}$ of the classification of Terry Wall defined by the equations

$$xy + z^{l-1} = 0$$
$$xz + yz^2 + y^{k-1} = 0$$

$$4 \leq l \leq k, 5 \leq k.$$

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- The Poincaré complex of $(X, 0)$ is exact.

μ and τ

Let $(X,0)$ be a germ of a space curve singularity defined by $f=g=0$, with $f,g\in\mathbb{C}\{x,y,z\}$

- $\mu(X,0)=dim_{\mathbb{C}}(\Omega^1_{X,0}/d\mathcal{O}_{(X,0)})$
- $\tau(X,0)=dim_{\mathbb{C}}(\mathbb{C}\{x,y,z\}/<f,g,M_1,M_2,M_3>)$

  here the M_i are the 2-minors of the Jacobian matrix of f,g.

Poincaré complex

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$$4\le l\le k, 5\le k.$$

The Poincaré complex

$$0\longrightarrow\mathbb{C}\longrightarrow\mathcal{O}_{X_{l,k},0}\longrightarrow\Omega^1_{X_{l,k},0}\longrightarrow\Omega^2_{X_{l,k},0}\longrightarrow\Omega^3_{X_{l,k},0}\longrightarrow 0$$

is exact.
But $(X_{l,k},0)$ is not quasi-homogeneous:

$$\mu(X,0)=\tau(X,0)+1=k+l+2.$$

μ and τ

Let $(X,0)$ be a germ of a space curve singularity defined by $f=g=0$, with $f,g\in\mathbb{C}\{x,y,z\}$

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  here the M_i are the 2-minors of the Jacobian matrix of f,g.
- Reiffen: The Poincaré complex is exact if and only if

$$<f,g>\Omega^3_{\mathbb{C}^3,0}\subset d(<f,g>\Omega^2_{\mathbb{C}^3,0})$$

and

$$\mu(X,0)=dim_{\mathbb{C}}(\Omega^2_{X,0})-dim_{\mathbb{C}}(\Omega^3_{X,0})$$

μ and τ

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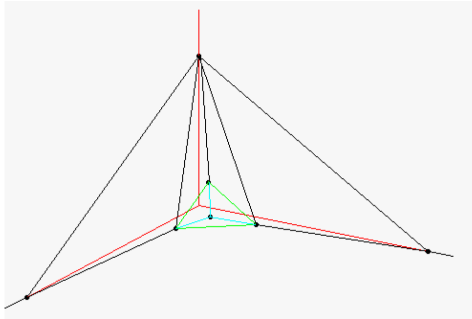
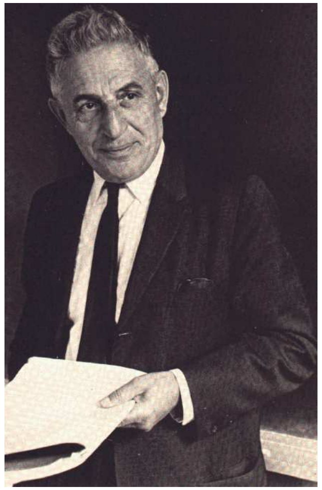
- $\mu(X,0)=dim_{\mathbb{C}}(\Omega^1_{X,0}/d\mathcal{O}_{(X,0)})$

Zariski's conjecture

$$F_t = x^a + y^b + z^{3c} + x^{c+2}y^{c-1} + x^{c-1}y^{c-1}z^3 + x^{c-2}y^c(y^2 + tx)^2$$



Zariski's conjecture



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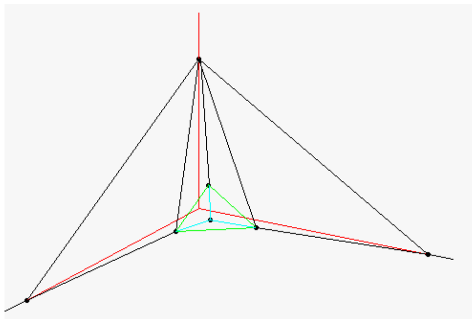
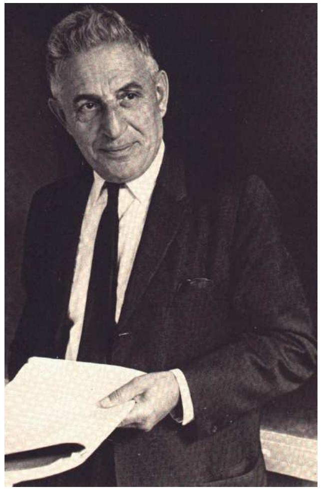


$(a, b, c) = (40, 30, 8)$
$\mu(F_0) = 10661$
$\mu(F_t) = 10655$

Zariski's conjecture



Conjecture (Zariski 1971) :
A μ–constant deformation of an isolated hypersurface singularity is a deformation with constant multiplicity.

nilpotent Groups

Let G be a finite group

$$G^{(1)} := [G, G] = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle .$$

Let $G^{(i)} := [G^{(i-1)}, G]$, then G is called nilpotent, if $G^{(m)} = \{e\}$ for some m.

Computeralgebra and finite Groups

Problem: Characterize the class of finite solvable groups G by 2–variable identities.

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- abelian groups are nilpotent.
- groups of order of a prime power are nilpotent.
- G is nilpotent $\Leftrightarrow$ it is a direct product of its Sylow groups.
- S_3 is not nilpotent.

Computeralgebra and finite Groups

Problem: Characterize the class of finite solvable groups G by 2–variable identities.

Example:

- G is **abelian** $\Leftrightarrow xy = yx \;\forall\, x, y \in G$
- (Zorn, 1930) A finite group G is **nilpotent** $\Leftrightarrow \exists\, n \geq 1$, such that $v_n(x, y) = 1 \;\forall\, x, y \in G$
  (**Engel Identity**)

  $v_1 := [x, y] = xyx^{-1}y^{-1}$ (commutator)
  $v_{n+1} := [v_n, y]$

Main result

Theorem (T. Bandman, G.-M. Greuel, F. Grunewald, B. Kunyavsky, G. Pfister, E. Plotkin)

$$U_1 = U_1(x,y) := x^2y^{-1}x,$$
$$U_{n+1} = U_{n+1}(x,y) = [xU_nx^{-1}, yU_ny^{-1}].$$

A finite group G is **solvable** $\Leftrightarrow \exists\, n$, such that $U_n(x,y) = 1\ \forall\, x,y \in G$.

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A finite group G is **solvable** $\Leftrightarrow \exists\, n$, such that $U_n(x,y) = 1\ \forall\, x,y \in G$.

- $U_1(x,y) = 1 \Leftrightarrow y = x^{-1}$
- $U_1(x,y) = U_2(x,y)$
  $\Leftrightarrow x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}$
- Let $x, y \in G$ such that $y \neq x^{-1}$ and
  $U_1(x,y) = U_2(x,y) \Rightarrow U_n(x,y) \neq 1\ \forall\, n \in \mathbb{N}$.

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solvable Groups

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$$G^{(i)} := [G^{(i-1)}, G^{(i-1)}],$$

then G is called solvable, if $G^{(m)} = \{e\}$ for some m.

- nilpotent groups are solvable.
- S_3, S_4 are solvable.
- groups of odd order are solvable.
- S_5, A_5 are not solvable.

Proof

G solvable $\Rightarrow$ Identity is true (by definition).
Idea of $\Leftarrow$
Theorem (Thompson, 1968)
Let G minimally not solvable. Then G is one of the following groups:

- **PSL**$(2, \mathbb{F}_p)$, p a prime number ≥ 5

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- **PSL**$(3, \mathbb{F}_3)$
- **Sz**(2^p) p a prime number.

If is enough to prove (for G in Thompson's list): $\exists\, x, y \in G$, such that $y \neq x^{-1}$ and $U_1(x, y) = U_2(x, y)$.

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PSL

$$\text{PSL}(2, K) = \text{SL}(2, K) / \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} \right) \,\middle|\, a^2 = 1 \right\}$$

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Motivation of the Choice of the Word

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Let w be a word in X, Y, X^{-1}, Y^{-1} and

$$U_1 = w$$
$$U_{n+1} = [XU_nX^{-1}, YU_nY^{-1}].$$

PSL

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$$\text{PSL}(2,K) = \text{SL}(2,K)/\left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \;\middle|\; a^2 = 1\right\}$$

especially

$$\text{PSL}(2,\mathbb{F}_5) = \{[\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)],\ a_{11}a_{22} - a_{21}a_{12} = 1\}$$
$$[\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)] = \left\{\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right), \left(\begin{smallmatrix} 4a_{11} & 4a_{12} \\ 4a_{21} & 4a_{22} \end{smallmatrix}\right)\right\}.$$

Motivation of the Choice of the Word

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Let w be a word in X, Y, X^{-1}, Y^{-1} and

$$U_1 = w$$
$$U_{n+1} = [XU_nX^{-1}, YU_nY^{-1}].$$

A computer–search through the 10,000 shortest words in X, X^{-1}, Y, Y^{-1} found the following four words, such that the equation $U_1 = U_2$ has a non-trivial solution in PSL$(2,p)$ for all $p < 1000$:

$$w_1 = X^{-2}Y^{-1}X$$
$$w_2 = X^{-1}YXY^{-1}X$$
$$w_3 = Y^{-2}X^{-1}$$
$$w_4 = XY^{-2}X^{-1}YX^{-1}$$

PSL

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It holds:

$$\text{PSL}(2,\mathbb{F}_5) \cong \text{PSL}(2,\mathbb{F}_4) \cong A_5$$

Translation to algebraic Geometry

Let us consider $G = \mathbf{PSL}(\mathbf{2}, \mathbb{F}_{\boldsymbol{p}}),\ \mathbf{p} \geq \mathbf{5}$

Consider the matrices

$$x = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix}$$

$x^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & t \end{smallmatrix}\right)$ implies $y \neq x^{-1}$ for all $(b,c,t) \in \mathbb{F}_p^3$.

It is enough to prove that the equation

$$U_1(x,y) = U_2(x,y), \text{ i.e.}$$

$$x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}$$

has a solution $(b,c,t) \in \mathbb{F}_p^3$.

Translation to algebraic Geometry

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The equations

The entries of $U_1(x,y) - U_2(x,y)$ are the following polynomials in $\mathbb{Z}[b,c,t]$:

$$\begin{aligned}
p_1 = \;& b^3c^2t^2 + b^2c^2t^3 - b^2c^2t^2 - bc^2t^3 - b^3ct + b^2c^2t + b^2ct^2 + 2bc^2t^2 \\
& + bct^3 + b^2c^2 + b^2ct + bc^2t - bct^2 - c^2t^2 - ct^3 - b^2t + bct + c^2t \\
& + ct^2 + 2bc + c^2 + bt +^2 ct + c + 1 \\
p_2 = \;& -b^3ct^2 - b^2ct^3 + b^2c^2t + bc^2t^2 + b^3t - b^2ct - 2bct^2 - b^2c + bct \\
& + c^2t + ct^2 - bt - ct - b - c - 1 \\
p_3 = \;& b^3c^3t^2 + b^2c^3t^3 - b^2c^2t^3 - bc^2t^4 - b^3c^2t + b^2c^3t +^2 b^2c^2t^2 \\
& + 2bc^3t^2 +^2 bc^2t^3 + b^2c^2t +^2 b^2ct^2 + bc^2t^2 - c^2t^3 - ct^4 - 2b^2ct \\
& + bc^2t + c^3t + bct^2 + 2c^2t^2 + ct^3 - b^2c - b^2t + bct + c^2t + bt^2 \\
& + 3ct^2 + bc - bt - b - c + 1 \\
p_4 = \;& -b^3c^2t^2 - b^2c^2t^3 + b^2c^2t^2 + bc^2t^3 + b^3ct - b^2c^2t - b^2ct^2 - 2bc^2t^2 \\
& - bct^3 - 2b^2ct + c^2t^2 + ct^3 + b^2t - bct - c^2t - ct^2 + b^2 - bt \\
& - 2ct - b - t + 1
\end{aligned}$$

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Consider the matrices

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Hasse–Weil–Theorem

Theorem von Hasse–Weil (generalized by Aubry and Perret for singulare curves):
Let $C \subseteq \mathbb{A}^n$ be an absolutely irreducible affine curve defined over the finite field $\mathbb{F}_q$ and $\overline{C} \subset \mathbb{P}^n$ its projective closure $\Rightarrow$

$$\#C(\mathbb{F}_q) \geq q + 1 - 2p_a\sqrt{q} - d$$

(d = degree, p_a = arithmetic genus of $\overline{C}$).

The Hilbert–polynomial of $\overline{C}$, $H(t) = d \cdot t - p_a + 1$, can be computed using the ideal I_h of $\overline{C}$:
We obtain $H(t) = 10t - 11 \Rightarrow d = 10,\ p_a = 12$.

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We obtain $H(t) = 10t - 11 \Rightarrow d = 10,\ p_a = 12$.
Since $p + 1 - 24\sqrt{p} - 10 > 0$ if $p > 593,$ we obtain the result.

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absolute irreduciblity

Proposition: $V(I^{(p)})$ is absolutely irreduzibel for all primes $p \geq 5$.
Beweis:

Using **SINGULAR** we prove:

$$\langle f_1, f_2 \rangle : h^2 = I.$$

$$\begin{aligned} f_1 = \ & t^2 b^4 + (t^4 - 2t^3 - 2t^2) b^3 - (t^5 - 2t^4 - t^2 - 2t - 1) b^2 \\ & -(t^5 - 4t^4 + t^3 + 6t^2 + 2t) b + (t^4 - 4t^3 + 2t^2 + 4t + 1) \\ f_2 = \ & (t^3 - 2t^2 - t) c + t^2 b^3 + (t^4 - 2t^3 - 2t^2) b^2 \\ & -(t^5 - 2t^4 - t^2 - 2t - 1) b - (t^5 - 4t^4 + t^3 + 6t^2 + 2t) \\ h = \ & t^3 - 2t^2 - t \end{aligned}$$

absolute irreduciblity

Proposition: $V(I^{(p)})$ is absolutely irreduzibel for all primes $p \geq 5$.

We give explicitely matrices M and N with entries in $\mathbb{Z}[b, c, t]$ such

that $M \begin{pmatrix} p_1 \\ \vdots \\ p_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $N \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} h^2 p_1 \\ \vdots \\ h^2 p_4 \end{pmatrix}$

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Step 2

f_2 is linear in c , it is enough to show, that f_1 is absolutely irreducibel.

algebraically the following is equivalent:

- $IK[b, c, t]$ is prime
- $\langle f_1, f_2 \rangle K(t)[b, c]$ prime
- f_1 irreducibel in $K(t)[b]$ resp. in $K[t, b]$.

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We obtain for all fields K

$$IK[b, c, t] = \big(\langle f_1, f_2 \rangle K[b, c, t]\big) : h^2 \,.$$

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geometrically:
Curve $V(I)$ is irreducibel, if the projection to the b, t–plane is irreducibel.

Step 2

f_2 is linear in c , it is enough to show, that f_1 is absolutely irreducibel.

Ansatz

$$(*) \qquad P = (x^2 + ax + b)(x^2 + gx + d)$$

a, b, g, d polynomials in t with variable coefficients

$$a(i),\ b(i),\ g(i),\ d(i)\,.$$

Let $P(x) := t^2 J[1]|_{b=x/t}$ then P is monic of degree 4.

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a, b, g, d polynomials in t with variable coefficients

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The decomposition $(*)$ with `a(i), b(i), g(i), d(i)` $\in \overline{\mathbb{F}}_p$ does not exist iff the ideal `C` generated by the coefficients with respect to x, t of $P - (x^2 + ax + b)(x^2 + gx + d)$ has no solution in $\overline{\mathbb{F}}_p$. This is equivalent to the fact that $1 \in$ `C`.

Let $P(x) := t^2 J[1]|_{b=x/t}$ then P is monic of degree 4.

$x^4 + (t^3 - 2t^2 - 2t)x^3 - (t^5 - 2t^4 - t^2 - 2t - 1)x^2 -$
$(t^6 - 4t^5 + t^4 + 6t^3 + 2t^2)x + (t^6 - 4t^5 + 2t^4 + 4t^3 + t^2)$.

We prove, that the induced polynomial $P \in \mathbb{F}_p[t, x]$ is absolutely irreducibel for all primes $p \geq 2$.
(Using the lemma of Gauß this is equivalent to P being irreducibel in $\overline{\mathbb{F}}_p(t)[x]$.)

PSL: The case $q = p = 5, \ldots, 593$

p	Pts. in $V^{(p)}$	p	Pts. in $V^{(p)}$	p	Pts. in $V^{(p)}$	p	Pts. in $V^{(p)}$
5	(1,2,2)	113	(0,37,52)	269	(2,205,73)	433	(0,67,228)
7	(0,1,4)	127	(0,10,112)	271	(0,64,97)	439	(0,4,22)
11	(1,9,1)	131	(1,14,22)	277	(4,21,7)	443	(2,213,143)
13	(1,1,8)	137	(0,5,32)	281	(0,98,150)	449	(2,215,286)
17	(0,7,7)	139	(1,19,109)	283	(1,188,250)	457	(0,63,378)
19	(3,2,10)	149	(1,87,63)	293	(1,26,270)	461	(5,5,267)
23	(0,11,19)	151	(1,99,108)	307	(1,100,10)	463	(0,62,204)
29	(2,12,8)	157	(1,22,62)	311	(2,56,162)	467	(1,70,461)
31	(1,18,26)	163	(1,67,8)	313	(0,45,194)	479	(0,202,293)
37	(1,25,22)	167	(0,3,14)	317	(2,34,146)	487	(0,9,92)
41	(1,4,19)	173	(1,101,119)	331	(1,197,323)	491	(1,31,439)
43	(1,15,3)	179	(1,11,71)	337	(0,138,312)	499	(1,275,40)
47	(0,2,8)	181	(1,3,75)	347	(1,252,267)	503	(0,12,158)
53	(2,16,12)	191	(0,7,58)	349	(2,314,255)	509	(7,424,256)
59	(3,33,39)	193	(0,45,142)	353	(0,142,187)	521	(0,219,250)
61	(2,21,49)	197	(1,18,145)	359	(0,80,20)	523	(3,8,369)
67	(1,11,63)	199	(0,67,180)	367	(0,28,80)	541	(1,220,80)
71	(0,18,60)	211	(1,51,92)	373	(1,82,336)	547	(2,264,122)
73	(1,44,49)	223	(5,6,157)	379	(2,9,197)	557	(2,42,261)
79	(0,17,71)	227	(1,118,74)	383	(0,149,138)	563	(1,317,485)
83	(1,54,39)	229	(3,220,92)	389	(1,27,379)	569	(0,269,369)
89	(0,19,26)	233	(0,19,149)	397	(3,271,169)	571	(1,443,422)
97	(0,10,15)	239	(1,179,126)	401	(0,48,349)	577	(2,169,514)
101	(2,1,47)	241	(0,67,220)	409	(0,50,98)	587	(1,45,229)
103	(0,23,39)	251	(3,15,112)	419	(1,121,65)	593	(1,240,5).
107	(1,61,26)	257	(3,97,135)	421	(2,331,151)		
109	(1,69,102)	263	(0,47,154)	431	(0,100,189)		

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The ideal of the coefficients of `C`:

```
C[1]=-b(5)*d(3)
C[2]=-b(5)*g(2)
C[3]=-b(4)*d(3)-b(5)*d(2)
C[4]=-b(4)*g(2)-b(5)*g(1)-d(3)-1
C[5]=-b(3)*d(3)-b(4)*d(2)-b(5)*d(1)+1
C[6]=-b(5)-g(2)-1
C[7]=a(0)*b(5)-a(2)*d(3)-b(3)*g(2)-b(4)*g(1)-d(2)+4
C[8]=-a(0)^2*b(5)+b(0)*b(5)-b(2)*d(3)-b(3)*d(2)-b(4)*d(1)-b(5)-4
C[9]=-a(2)*g(2)-b(4)-g(1)+2
C[10]=a(0)*b(4)-a(1)*d(3)-a(2)*d(2)-b(2)*g(2)-b(3)*g(1)-d(1)-1
C[11]=-a(0)^2*b(4)+b(0)*b(4)-b(1)*d(3)-b(2)*d(2)-b(3)*d(1)-b(4)+2
C[12]=a(0)-a(1)*g(2)-a(2)*g(1)-b(3)-d(3)
C[13]=-a(0)^2+a(0)*b(3)-a(0)*d(3)-a(1)*d(2)-a(2)*d(1)+b(0)-b(1)*g(2)-b(2)*g(1)-7
C[14]=-a(0)^2*b(3)+b(0)*b(3)-b(0)*d(3)-b(1)*d(2)-b(2)*d(1)-b(3)+4
C[15]=-a(2)-g(2)-2
C[16]=a(0)*a(2)-a(0)*g(2)-a(1)*g(1)-b(2)-d(2)+1
C[17]=-a(0)^2*a(2)+a(0)*b(2)-a(0)*d(2)-a(1)*d(1)+a(2)*b(0)-a(2)-b(0)*g(2)-b(1)*g(1)-2
C[18]=-a(0)^2*b(2)+b(0)*b(2)-b(0)*d(2)-b(1)*d(1)-b(2)+1
C[19]=-a(1)-g(1)-2
C[20]=a(0)*a(1)-a(0)*g(1)-b(1)-d(1)+2
C[21]=-a(0)^2*a(1)+a(0)*b(1)-a(0)*d(1)+a(1)*b(0)-a(1)-b(0)*g(1)
C[22]=-a(0)^2*b(1)+b(0)*b(1)-b(0)*d(1)-b(1)
C[23]=-a(0)^3+2*a(0)*b(0)-a(0)
C[24]=-a(0)^2*b(0)+b(0)^2-b(0)
```


PSL: The cases $q = 2^n$ and $q = 3^n$

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n	point in $V^{(q)}$
2	$(a, 0, 1)$
3	(a, a^2, a^2)
4	(a^3, a^{12}, a^5)
5	(a^3, a^{20}, a^{22})
6	(a^9, a^9, a^{54})
7	(a, a^{62}, a^{48})
8	(a, a^{70}, a^{200})
9	(a, a^{191}, a^{121}).

n	point in $V^{(q)}$
2	$(a, 0, a)$
3	(a, a^3, a^{10})
4	$(a, -1, a^{66})$
5	(a^2, a^{10}, a^2).

Schritt 2

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Using SINGULAR, one shows that over $\mathbb{Z}\big[\{a(i)\},\ \{b(i)\},\ \{g(i)\},\ \{d(i)\}\big]$

$$4 = \sum_{i=1}^{24} M_i \, \mathtt{C}[i]\,.$$

Suzuki groups

This case is much more complicated.
We have to prove that on a surface U any odd power of a certain endomorphism θ has fixed points.

PSL(3,3)

Die Gruppe PSL(3,3)

One easily checks that $x = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&1 \end{pmatrix}$ und $y = \begin{pmatrix} 2&0&2\\0&1&1\\2&1&1 \end{pmatrix}$

$$x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}$$

i.e. for

Suzuki groups

This case is much more complicated.
We have to prove that on a surface U any odd power of a certain endomorphism θ has fixed points.
Here we use the **Lefschetz–Weil–Grothendieck trace formulae** generalized by Deligne–Lusztig, Th. Zink, Pink, Katz and Adolphson–Sperber:

$$2^n - b_1(U) \cdot 2^{\frac{3}{4}n} - b_2(U) \cdot 2^{\frac{1}{2}n} \leq \#\,\text{Fix}\,(\theta^n, U)$$

for n sufficientely large.

PSL(3,3)

Die Gruppe PSL(3,3)

One easily checks that $x = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&1 \end{pmatrix}$ und $y = \begin{pmatrix} 2&0&2\\0&1&1\\2&1&1 \end{pmatrix}$

$$x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}$$

i.e. for

$$w = x^2y^{-1}x$$

and

$$U_1 = w,\ U_2 = [xU_1x^{-1}, yU_1y^{-1}]$$

holds

$$U_1(x,y) = U_2(x,y).$$

Suzuki Groups

$$M(c) = \begin{pmatrix} c^{1+2^m} & 0 & 0 & 0 \\ 0 & c^{2^m} & 0 & 0 \\ 0 & 0 & c^{-2^m} & 0 \\ 0 & 0 & 0 & c^{-1-2^m} \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Suzuki Groups

Let $n = 2m+1$, $q = 2^n$ and consider the automorphism

$$\pi : \mathbb{F}_q \longrightarrow \mathbb{F}_q, \qquad \pi(a) = a^{2^{m+1}}.$$

Note: π^2 is the Frobenius.

Suzuki Groups

Recall,

$$U_1(x,y) = U_2(x,y)$$

if and only if

$$x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}.$$

Suzuki Groups

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Note: π^2 is the Frobenius.

$$\mathrm{Sz}(q) = \langle U(a,b), M(c), T \mid a,b,c \in \mathbb{F}_q, c \neq 0 \rangle$$

$$U(a,b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a\pi(a)+b & \pi(a) & 1 & 0 \\ a^2\pi(a)+ab+\pi(b) & b & a & 1 \end{pmatrix}$$

Aim: We show that the variety $V(n) \subset \mathbb{F}_q^4$ is not empty.

Problem: We cannot work with infinetely many systems of equations.
To be independent on n we replace the expressions $\pi(a), \pi(b), \pi(c), \pi(d)$ by the variables a_0, b_0, c_0, d_0.

$$S(a,b,a_0,b_0) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ aa_0+b & a_0 & 1 & 0 \\ a^2a_0+ab+b_0 & b & a & 1 \end{pmatrix}$$

then

$$U(a,b) = S\big(a,b,\pi(a),\pi(b)\big).$$

Recall,

$$U_1(x,y) = U_2(x,y)$$

if and only if

$$x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}.$$

We fix two matrices

$$x = TU(a,b), y = TU(c,d) \in \text{Sz}(q).$$

The equations of the variety $V(n)$ defined by $U_1 = U_2$ depend on n $(q = 2^n)$.

We consider the matrices

$$x = TS(a,b,a_0,b_0)$$
$$y = TS(c,d,c_0,d_0)$$

and obtain from

$$U_1(x,y) = U_2(x,y)$$

a system of equations, defining a variety $V \subset \mathbb{F}_2^8$, not depending on n.

Aim: We show that the variety $V(n) \subset \mathbb{F}_q^4$ is not empty.

Problem: We cannot work with infinetely many systems of equations.
To be independent on n we replace the expressions $\pi(a), \pi(b), \pi(c), \pi(d)$ by the variables a_0, b_0, c_0, d_0.

Suzuki Groups

On $V \subset \mathbb{F}_2^8$ we consider the endomorphism

$$\theta : V \longrightarrow V$$
$$\theta(a,b,c,d,a_0,b_0,c_0,d_0) = (a_0,b_0,c_0,d_0,a^2,b^2,c^2,d^2).$$

Then is θ^2 the Frobenius.

Suzuki Groups

The ideal of an irreducible component of V:

```
J[1]=d2+adv+cdv+a2v2+c2v2+abx+bcx+wx+c2x2+vy+xy+c2;
J[2]=a2b+acd+a2cv+aw+a3x+a2cx+ac2x+ay+av+cx;
J[3]=bcw+acvw+w2+a2wx+acwx+b2+bd+d2+abv+bcv+c2v2+bcx+adx+a4+a3c+vx+x2+ac+1;
J[4]=adv2+cdv2+d2x+abvx+bcvx+advx+cdvx+vwx+abx2+bcx2+wx2+c2x3+v2y+vxy+x2y+ab+cd+acv
     +c2v+w+a2x+acx+c2x+y;
J[5]=abd+abcv+bc2v+a2dv+dw+avw+cvw+bc2x+c2dx+ac2vx+awx+a2cx2+ac2x2+c3x2+by+cxy+dv+av2
     +cv2+bx+cx2+ac2+a+c;
J[6]=bcd+cd2+a2bv+abcv+a2dv+c2dv+bw+avw+cvw+a2dx+c2dx+c3vx+a3x2+a2cx2+ac2x2+by+dy+cvy
     +axy+bv+dv+cv2+dx+cvx+ax2+a3+a+c;
J[7]=a3v2+a2cv2+c2dx+a3vx+ac2vx+a2cx2+ac2x2+c3x2+cxy+cx2;
J[8]=d2v+acv3+c2v3+cdvx+a2vx2+acvx2+a2bc+ac2d+ac3v+acw+a3cx+vx2+acy+a2v+acx+v;
J[9]=advx+cdvx+a2v2x+c2v2x+abx2+bcx2+a2vx2+c2vx2+wx2+vxy+c3d+a3cv+a2c2v+a3cx+a2c2x+c4x
     +c2y+cd+a2v+c2v+c2x+y;
J[10]=a2vw+acvw+c2vw+w2+ac2dx+c3dx+a3cvx+ac3vx+acwx+c2wx+a3cx2+c4x2+aby+acxy+c2xy+a2v2
      +acv2+abx+adx+cdx+a2vx+acvx+c2vx+a2x2+c2x2+a4+a2c2+v2+1;
```


Suzuki Groups

On $V \subset \mathbb{F}_2^8$ we consider the endomorphism

$$\theta : V \longrightarrow V$$
$$\theta(a,b,c,d,a_0,b_0,c_0,d_0) = (a_0,b_0,c_0,d_0,a^2,b^2,c^2,d^2).$$

Then is θ^2 the Frobenius.

The following holds for $p = (a,b,c,d) \in \overline{\mathbb{F}}_2^4$:

$p \in V(n) \subset \mathbb{F}_q^4$

$$\Updownarrow$$

$$\begin{array}{lllll}
(1) & (a,b,c,d,a^{2^{m+1}}, & b^{2^{m+1}}, & c^{2^{m+1}}, & d^{2^{m+1}}) \in V \\
 & \qquad \| & \| & \| & \| \\
 & \qquad \pi(a) & \pi(b) & \pi(c) & \pi(d)
\end{array}$$

(2) $a^q = a, \dots, d^q = d$, d.h. $a, \dots, d \in \mathbb{F}_q$.

We use this property to obtain $V(n)$ as fixed point set of the n–th power of θ in V .

Suzuki Groups

As in the PSL(2)–Fall, we will prove, that this componet is absolutey irreducible.

Problem: V is a surface and the equations are much more complicated.

Suzuki Gruppen

We show that θ^n has fixed points for all odd n .

Suzuki Groups

Let $(a, b, c, d, a_0, b_0, c_0, d_0) \in V \subset \overline{\mathbb{F}}_2^8$ and
$\theta^n(a, \ldots, d_0) = (a, \ldots, d_0) \qquad (n = 2m+1)$
$\Rightarrow$

$$\begin{aligned} a_0 = a^{2^{m+1}} &= \pi(a), \ldots, d_0 = d^{2^{m+1}} = \pi(d) \\ a = a^q, &\ldots, d^q = d. \end{aligned}$$

Suzuki Gruppen

We show that θ^n has fixed points for all odd n . To prove that θ^n has fixed points we use the **Lefschetz–Weil–Grothendieck trace formulae** generalized by Deligne–Lusztig, Th. Zink, Pink, Katz and Adolphson–Sperber:

We obtain an affine, open, smooth and invariant sub-set U of V, such that:

$$\left| \text{Fix } (\theta^n, U) - 2^n \right| \leq b_1(U) \cdot 2^{\frac{3}{4}n} + b_2(U) \cdot 2^{\frac{1}{2}n}$$

for n sufficientely large.

Suzuki Groups

Let $(a, b, c, d, a_0, b_0, c_0, d_0) \in V \subset \overline{\mathbb{F}}_2^8$ and
$\theta^n(a, \ldots, d_0) = (a, \ldots, d_0) \qquad (n = 2m+1)$
$\Rightarrow$

$$\begin{aligned} a_0 = a^{2^{m+1}} &= \pi(a), \ldots, d_0 = d^{2^{m+1}} = \pi(d) \\ a = a^q, &\ldots, d^q = d. \end{aligned}$$

We obtain:

If $(a, b, c, d, a_0, b_0, c_0, d_0) \in V$ is a fixed point of θ^n , then $(a, b, c, d) \in V(n)$.

Problem: We have to show that for all primes n, θ^n has fixed points in V .

Suzuki Groups

For the Betti numbers b_1 and b_2 we obtain

$$b_1(U) < 2^9, \quad b_2(U) < 2^{23}.$$

To obtain fixed points we need

$$2^n > 2^9 \cdot 2^{\frac{3}{4}n} + 2^{23} \cdot 2^{\frac{n}{2}},$$

which is true for $n > 52$.