Applying Computer Algebra in Mathematical Research: An Example from Group Theory

The solution to this problem may serve as a model for how computer algebra methods may be used for establishing conjectures and for proving theorems in other fields of mathematics.

> *Characterize the class of solvable finite groups G by explicit two-variable identities.*

To explain this problem, note that a group G is Abelian iff the two-variable identity $xy = yx$ is satisfied for all $x, y \in G$. Moreover, Zorn (1930) proved that, setting

$$v_1(x,y) := \big[x,y\big] := xyx^{-1}y^{-1}\,, \qquad v_{k+1}(x,y) := \big[v_k, y\big]\,,$$

a finite group G is nilpotent iff there exists some $n \geq 1$ such that the two-variable identity $v_n(x,y) = 1$ holds for all $x, y \in G$.

Theorem. *Define U_k inductively by*

$$U_1(x,y) := x^{-2}y^{-1}x, \qquad U_{k+1}(x,y) := \big[xU_k(x,y)x^{-1}, yU_k(x,y)y^{-1}\big]\,.$$

Then a finite group G is solvable iff there exist some n such that the two-variable identity $U_n(x,y) = 1$ holds for all $x, y \in G$.

That solvable groups satisfy the identity above is clear by the definition of a solvable group. Thus, it remains to show that for a (minimal) non-solvable finite group no such equality holds. Fortunately, the minimal non-solvable finite groups have been classified by Thompson (1968): his list consists of

1. $\mathrm{PSL}(2, \mathbb{F}_p)$, $\quad p \geq 5$ prime,
2. $\mathrm{PSL}(2, \mathbb{F}_{2^p})$, $\quad p$ prime,
3. $\mathrm{PSL}(2, \mathbb{F}_{3^p})$, $\quad p$ prime,
4. $\mathrm{PSL}(3, \mathbb{F}_3)$,
5. the Suzuki groups $\mathrm{Sz}(2^p)$, p prime.

The key observation that allows one to translate the problem to a problem of algebraic geometry is the following: *if $x, y \in G$ satisfy $1 \neq U_1(x,y) = U_2(x,y)$, then $U_n(x,y) \neq 1$ for all $n \in \mathbb{Z}$.*

It thus remains to show that for each group in Thompson's list, there are elements $x, y \in G$ such that $1 \neq U_1(x,y) = U_2(x,y)$.

It is quite instructive to show how such a problem from group theory can be translated to a problem in algebraic geometry and how to solve it with the help of computer algebra.