Drinfeld Modules and Elliptic Sheaves

Thesis — revised english version

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Contents

1 Introduction

Drinfeld Modules

Let X be a geometrical connected smooth algebraic curve over the finite field $\mathbb{F}_q$, let $\infty \in X$ be a closed point and let $A := \Gamma(X \setminus \infty, \mathcal{O}_X)$ be the ring of regular functions outside ∞. In this case A is a Dedekind ring. If we choose appropriate element $T \in A$ then A is a finite $\mathbb{F}_q[T]$-algebra and is the integral closure of $\mathbb{F}_q[T]$ in F, the function field of the curve X.
Let $\mathbb{C}_\infty$ be the completion of the algebraic closure of the completion of F at the point ∞. An A-lattice of rank d in $\mathbb{C}_\infty$ is a finitely generated discrete A-submodul in $\mathbb{C}_\infty$. A discription of the equivalence classes of these lattices modulo the $\mathbb{C}_\infty^\times$-action is given by the moduli-space of Drinfeld modules. Cf. *Elliptic Modules I/II* [Dri76] and [Dri77].
A Drinfeld module (over $\mathbb{C}_\infty$) is a non-trivial ringhomomorphism e from A into the ring of algebraic, $\mathbb{F}_q$-linear endomorphisms of the additive group $\mathbb{G}_{a/\mathbb{C}_\infty}$. Let $\partial : \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathbb{C}_\infty}) \longrightarrow \mathbb{C}_\infty$ be the canonical map. Additionally we require that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;e\;} & \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathbb{C}_\infty}) \\ & \mathrm{char} \searrow \quad \swarrow \partial & \\ & \mathbb{C}_\infty & \end{array}$$

commutes. In this case the map char is the canonical inclusion. In other words, a Drinfeld module is a new A-module structure on the additive group $(\mathbb{C}_\infty, +)$.
Let $\mathbb{C}_\infty\{\tau\}$ be the skew polynomial ring over $\mathbb{C}_\infty$ with the relation $\tau\lambda = \lambda^q\tau$. Then there is a canonical isomorphism

$$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathbb{C}_\infty}) \cong \mathbb{C}_\infty\{\tau\}.$$

In particular a Drinfeld module is give by a ring homomorphism $e : A \longrightarrow \mathbb{C}_\infty\{\tau\}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;e\;} & \mathbb{C}_\infty\{\tau\} \\ & \mathrm{char} \searrow \quad \swarrow \tau = 0 & \\ & \mathbb{C}_\infty & \end{array}$$

commutes.

The map e forces for all $0 \neq a \in A$ a condition on the leading coefficent of $e(a)$: There exists a number $d \in \mathbb{N}$ such that

$$\deg_\tau e(a) = -d \deg(\infty)\infty(a).$$

The number d is called the rank of the Drinfeld module.
If E is a Drinfeld module over $\mathbb{C}_\infty$ of rank d and if $0 \neq I \subsetneq A$ is an ideal, then we get by

$$E[I] := \{x \in \mathbb{C}_\infty \mid e_a(x) = 0 \text{ für alle } x \in I\}$$

the subgroup *I-division points* of $(\mathbb{C}_\infty, +)$. It is

$$(I^{-1}/A)^d \simeq E[I].$$

The choice of an isomorphism is called a *level-I-structure*.
As in the case of elliptic curves we can enlarge the definition of a Drinfeld module, division points and level structures to the case of an arbitrary base scheme S (over $\mathbb{F}_q$).

Elliptic sheaves

In the definition of a Drinfeld module over a ring R we need only the function ring A and not the whole curve X. But the point ∞ gives information by the degree of the element $e(a)$. Using this idea we can construct so called *elliptic sheaves* on $X \times \operatorname{spec} R$. Important is the fact that the skew polynomial ring $R\{\tau\}$ is a projective $R \otimes_{\mathbb{F}_q} A$-module.
Drinfeld explains the construction of an elliptic sheaf corresponding to a Drinfeld module in the article *Commutative subrings of certain noncommutative rings* ([Dri86]) and he shows the equivalence of the corresponding categories. In the article *Varieties of Modules of F-Sheaves* ([Dri87]) he describes additionally the construction of level structures.
The content of this aritcle is a detailed describtion of the constructions in the articles of Drinfeld (loc. cit.) Additionally we will enlarge the constructions to an arbitrary base scheme S and we will explain level structures of elliptic sheaves inside and outside the characteristic.

Structuring

In *section 2* we will define and explain some base definition and methods of commutative algebra and algebraic geometry, as they are needed in this work. More often than not we will only reference the proofs

An important step are the theorems describing base change on paramterized curves and the behaveior of cohomolgy and global section functors. In addition we will describe a construction of vector bundles on curves.
The content of *section 3* is the definition of Drinfeld modules over an arbitrary base scheme. We follow here the work of T. Lehmkuhl ([Leh00]). Using the article of A. Blum and U. Stuhler *Drinfeld Modules and Elliptic Sheaves*([BS97] we describe whoto associate vector bundels to a Drinfeld module. They will turn out to be *elliptic sheaves*. In contrast we will use a Proj-construction for the vector bundles and will proof that both constructions coincide.
In *section 4* we explain in detail the definition of vector bundles of general typ following V.G. Drinfeld ([Dri86]) and proof some easy results.
Section 5 will be concerned with elliptischen sheaves. We will shows the equivalence of the categories of (standard) Drinfeld modules and elliptic sheaves which statisfies an additional condition.
We will define level structures of Drinfeld modules and elliptic sheaves in *section 6*. First we discuss a problem in the definition of leve structures in the case that the characteristic meets the points of the level. The problem will be solved in the case of an reduced base scheme. Next we will show the equivalence between Drinfeld modules with level structure and elliptic sheaves with level structure.

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2 Algebraic Curves and Vector Bundles

2.1 Algebraic Tools

All rings above are commutative and posses a unit element.

Definition 2.1
Let A be a ring and let R be an A-algebra. We call an element $x \in R$ integral *over A, if one of the following equivalent conditions are true:*

- *x is a zero of a normalized polynomial in $A[X]$.*
- *The subalgebra $A[x]$ of R is a finitely generated A-module.*
- *There exits a faithfull $A[x]$-module, finitely generated as an A-module.*

Proposition 2.2
1) If B is integral over A, then $B \otimes_A R$ is integral over R (cf. [Bou98], chapter V, § 1.1, proposition 5, page 307).

2) Let A be a ring, let $S \subset A$ be a multiplicative closed subset and let R be an A-algebra. If A' is the integral closure of A in R then $S^{-1}A'$ is the integral closure of $S^{-1}A$ in $S^{-1}R$ (cf. [Bou98], chapter V, § 1.5., proposition 16, page 314).

3) Let A be an integral closed domain, let $K = \mathrm{Quot}(A)$ and let L/K be a finite, separable K-algebra. If B is the integral closure of A in L, then B is contained in a finite generated A-module ([Bou98] chapter V, § 1.6., proposition 18, page 317).

4) Let K be a field and R be an integral closed K-algebra. Let L/K be a separable extension of K. If $L \otimes_K R$ is a domain, then $L \otimes_K R$ is integrally closed ([Bou98], chapter V, § 1.7., proposition 19, page 318).

5) Let L/K be an algebraic field extension. Let v be a valuation of K and let A be its valuation ring. Let B be the integral closure of A in L. Let

$$\begin{aligned}\mathcal{B} &:= \{\text{valuation rings } B_w \text{ of } L \text{ lying over } v\}\\ \mathcal{M} &:= \{\text{maximal ideals } \mathfrak{m} \text{ of } B\}.\end{aligned}$$

There is a bijection:

$$\begin{aligned}\mathcal{B} &\longrightarrow \mathcal{M} \\ B_w &\longmapsto \mathfrak{m}(B_w) \cap B\end{aligned} \qquad\qquad \begin{aligned}\mathcal{M} &\longrightarrow \mathcal{B} \\ \mathfrak{m} &\longmapsto B_{\mathfrak{m}}\end{aligned}$$

(cf. [Bou98], chapter VI, §8.6., proposition 6, page 427).

Definition 2.3
We call a ring A a Dedekind ring*, if one of the following equivalent conditions are true:*

- *A is a noetherian domain and the localization rings $A_{\mathfrak{p}}$ are principal ideal domains for all prime ideals $\mathfrak{p} \in \operatorname{spec} A$.*
- *A is a normal, noetherian domain of dimension 0 or 1.*

(cf. [Liu02], chapter 1, page 11 and chapter 4, definition 1.2, page 115)

Remark 2.4
If A is a Dedekind ring then all localization rings of A are Dedekind rings.

Proposition 2.5
Let A be a Dedekind ring and let M be an A-module. The module M is flat if and only if it is torsion free.

Proof Cf. [Liu02], chapter 1, corollary 2.14, page 11. □

Lemma 2.6
Let K be a field and let A be a K-algebra. Let R be another K-algebra and let $R \longrightarrow A \otimes_K R$ be the canonical inclusion.

1) $A \otimes_K R$ is a flat R-algebra.

2) If M is a flat $A \otimes_K R$-module, then M is a flat R-module.

Proof Cf. [Mat86], §3, page 46. □

Proposition 2.7
Let R be a reduced, noetherian ring and let M be a finitely generated R-module. If $\dim_{k(\mathfrak{p})} M(\mathfrak{p}) = n$ is constant for all $\mathfrak{p} \in \operatorname{spec} R$, then M is a projective R-module of rang n. We define $k(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $M(\mathfrak{p}) := M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$.

Proof We show that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{spec} R$. Let be $\mathfrak{p} \in \operatorname{spec} R$. According to [Mat86], §2, theorem 2.3 it exists a minimal system of generators $m_1, \ldots, m_n$ of $M_{\mathfrak{p}}$. We assume there is a non-trivial relation $\sum_{i=1}^{n} r_i m_i = 0$, whereas we set without loss of generality $r_1 \neq 0$. As well with R also $R_{\mathfrak{p}}$ is reduced, there is a prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$ with $r_1 \notin \mathfrak{q}$, and r_1 is a unit in $R_{\mathfrak{q}}$. So we conclude

$$m_1 = -\sum_{i=2}^{n} \frac{r_i}{r_1} m_i$$

in particular $m_2, \ldots, m_n$ is also a system of generators of $M_{\mathfrak{q}}$. This is a contradiction. □

Theorem 2.8
Let R be a ring, let I be a nilpotent ideal in R and let M be a R-module. The following conditions are equivalent:

1) M is flat as a R-module.

2) M/IM is flat as a R/I-module and $I \otimes_R M = IM$.

Proof Cf. [Mat86], §22, theorem 22.3, page 174. □

Theorem 2.9
Let M be a R-module. If M is projective then $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{spec} R$.

Proof Cf. [Eis94], exercise 4.11 a), page 136 and [Kap58], section 4, theorem 2, page 374. □

Proposition 2.10
Let M be a R-module of finite presentation. The module M is projective if and only if M is a flat R-module (projectivity always implies flat).

Proof Cf. [Eis94], chapter 6, exercise 6.2, page 172. □

Remark 2.11
In general a flat module is not projective. For example $\mathbb{Q}$ is a non-projective $\mathbb{Z}$-module.

Lemma 2.12
Let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of R-modules. If M and N are flat R-modules then L is a flat R-module (cf. [Wei94], chapter 3, exercise 3.2.2, page 69).

Proof Let P be a R-module. We tensorize the exact sequence with $\otimes_R P$. We get

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Tor}_1^R(L,P) \longrightarrow 0 \longrightarrow 0 \longrightarrow L \otimes P \longrightarrow \cdots.$$

From this follows $\operatorname{Tor}_1^R(L,P) = 0$. This proofs the assumption. □

2.2 Geometric Tools

Definition 2.13 (Affine varieties over a field)
Let K be a field. We define an affine variety *to be an affine scheme corresponding to a finitely generated K-algebra. We call a scheme X an* algebraic variety *over a field K if there exists a finite open covering of affine varieties. (cf. [Liu02], chapter 2, definition 3.47, page 55).*

Definition 2.14 (Regular points and regular varieties)
Let X be a local noetherian scheme. We call a point $x \in X$ regular *if the local ring $\mathcal{O}_{X,x}$ is regular. X itself is called* regular *if all points $x \in X$ are regular (cf. [Liu02], chapter 4, definition 2.8, page 128).*

Definition 2.15 (Smooth points and smooth varieties)
Let X be an algebraic variety over K and let K^{alg} be an algebraic closure of K. A point $x \in X$ is called smooth *if all points in the preimage of x in $X_{K^{\text{alg}}}$ are regular. X itself is called* smooth *if $X_{K^{\text{alg}}}$ is regular (cf. [Liu02], chapter 4, definition 3.28, page 141).*

Proposition 2.16
1) Let X be an algebraic variety over K. If $x \in X$ is smooth then x is regular (cf. [Liu02], chapter 4, corollary 3.32, page 142).

2) Let X be an algebraic variety over a perfect field K. X is smooth if and only if X is regular (cf. [Liu02], chapter 4, corollary 3.33, page 142).

Definition 2.17 (Smoothness over a local noetherian base scheme)
Let Y be a local noetherian scheme and let $f : X \longrightarrow Y$ be a morphism of finite type. The map f is called smooth *at the point $x \in X$ if the fiber $X_y \longrightarrow \operatorname{spec} k(y)$ is smooth. Here we set $y := f(x)$ (cf. [Liu02], chapter 4, definition 3.35, page 142).*

Proposition 2.18
Smooth morphisms are stable under base change, composition and fiber products (cf. [Liu02], chapter 4, proposition 3.38, page 143).

Definition 2.19 (Normal points)
Let X be a scheme. A point $x \in X$ is called normal *if $\mathcal{O}_{X,x}$ is an integral closed domain. X itself is called* normal *if X is irreducible and normal for all points $x \in X$ (cf. [Liu02] chapter 4, definition 1.1, page 115).*

Proposition 2.20

1) Let X be an algebraic variety over a field K and let L be a field extension of K. If X is geometrically reduced (or geometrically integral, - irreducible, - connected) then X_L is geometrically reduced (or geometrically integral, - irreducible, - connected).

2) Let X be a geometrically integral, algebraic variety over a field K. If Y is an integral variety over K then $X \times_K Y$ is again integral. The same statement is true for reduced, irreducible, connected.

Proof Cf. [Liu02], chapter 3, exercise 2.14, 2.15, page 97, 98. □

Lemma 2.21

Let $\iota : X \longrightarrow Y$ be a morphism of schemes. Let $\mathscr{F}$ be a quasi coherent sheaf of $\mathcal{O}_X$-modules and let $\mathscr{G}$ be a quasi coherent sheaf of $\mathcal{O}_Y$-modules.

1) If ι is an immersion[1] then there is a canonical isomorphism

$$\iota^*\iota_*\mathscr{G} \cong \mathscr{G}.$$

2) If $\iota : Y \longrightarrow X$ is an affine map then there is a canonical isomorphism

$$\mathscr{F} \otimes_{\mathcal{O}_X} \iota_*\mathcal{O}_Y \cong \iota_*\iota^*\mathscr{F}$$

3) If $\mathscr{I} \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathscr{I}\mathscr{F} = 0$ and if $\iota : Y \longrightarrow X$ is the related closed immersion then there is a canonical isomorphism $\mathscr{F} \cong \iota_*\iota^*\mathscr{F}$.

Proof The assertions 1) and 2) are proofed in [Liu02], chapter 5, exercise 1.1, page 171 f.
As a closed embedding is affine we can apply 2). We get

$$\begin{aligned} \iota_*\iota^*\mathscr{F} &\cong \mathscr{F} \otimes_{\mathcal{O}_X} \iota_*\mathcal{O}_Y \\ &\cong \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathscr{I} \\ &\cong \mathscr{F}/\mathscr{I}\mathscr{F} \\ &\cong \mathscr{F}. \end{aligned}$$

Lemma 2.22

[1] An immersion is a morphism which is an open immersion followed by a closed immersion

Let $f : X \longrightarrow S$ be a morphism of schemes and let $\mathscr{E}$ be a quasi coherent $\mathcal{O}_X$-module. Let $g : T \longrightarrow S$ be a base change. We obtain the following diagram

$$\begin{array}{ccc} X_T & \xrightarrow{\mathrm{pr}_X} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S. \end{array}$$

1) If f is separated and quasi compact and if g is a flat base change then there is a canonical isomorphism

$$g^* f_* \mathscr{E} \cong f_{T*} \operatorname{pr}_X^* \mathscr{E}.$$

2) In the setting of 1) let $\mathscr{F}$ be a locally free $\mathcal{O}_S$-module then we get the projection formula

$$f_* \mathscr{E} \otimes_{\mathcal{O}_S} \mathscr{F} \cong f_*(\mathscr{E} \otimes_{\mathcal{O}_X} f^* \mathscr{F}).$$

Proof Cf. [Liu02], chapter 5, exercise 1.16 b), c) page 174 and 175. □

2.3 Smooth and irreducible algebraic curves

Definition 2.23
Let K be a field. We call an algebraic variety X over K a curve *if all irreducible components are* 1*-dimensional. (cf. [Liu02], chapter 2, definition 5.29, page 75).*

Proposition 2.24
If X is an irreducible, smooth curve over a field K then for all affine, open subsets $U \subseteq X$ and for all points $x \in X$ the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,x}$ are Dedekind rings.

Proof If R is a regular, 0- or 1-dimensional local ring then R is a principal domain. The assertion follows now form the definition. □

Corollary 2.25
Let X be a geometrically irreducible, smooth curve over a field K. If L is a field extension of K then $X \times_K L$ is again a geometrically irreducible, smooth curve over L.

2.4 Projective algebraic curves

Below X is a smooth, projective, absolute irreducible curve over a field K. Let $\infty \in X$ be a closed point and let $\mathcal{O}_X(\infty)$ be the corresponding line bundle.

Proposition 2.26
If $P \in X$ is a closed point, then $X \setminus \{P\}$ is an affine, open subset of X.

Proof Cf. [Har], The Theorem of Riemann-Roch, page 248. □

Proposition 2.27
It is $H^0(X, \mathcal{O}_X) = K$.

Proof Cf. [Liu02], chapter 3, corollary 3.21, page 105. □

Proposition 2.28
The line bundle $\mathcal{O}_X(\infty)$ is ample.

Proof We have $\deg \mathcal{O}_X(\infty) = \deg(\infty) > 0$. The assumption follows now from [Liu02], chapter 7, proposition 5.5, page 305. □

Proposition 2.29
There exists a number $m \in \mathbb{N}$ such that $\mathcal{O}_X(\infty)^{\otimes m} \cong \mathcal{O}_X(m\infty)$ is very ample. In particular there is a closed embedding $f : X \longrightarrow \mathbb{P}^n_K$ such that

$$\mathcal{O}_X(m\infty) \cong f^*\mathcal{O}_{\mathbb{P}^n_K}(1).$$

Proof Cf. [Liu02], chapter 5, theorem 1.34, page 169. □

Proposition 2.30
Let $\mathscr{E}$ be a quasi-coherent $\mathcal{O}_X$-module on X. Let $\mathscr{E} := \mathscr{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n\infty)$ for all $n \in \mathbb{Z}$ and define $U := X \setminus \{\infty\}$. For all $n \in \mathbb{Z}$ let

$$H^0(X, \mathscr{E}(n\infty)) \longrightarrow H^0(U, \mathscr{E}(n\infty)) = H^0(U, \mathscr{E})$$

be the canonical maps and for all $n, m \in \mathbb{Z}$, $n \leq m$, let

$$H^0(X, \mathscr{E}(n\infty)) \longrightarrow H^0(X, \mathscr{E}(m\infty))$$

be the canonical inclusions. It follows

$$\varinjlim_n H^0(X, \mathscr{E}(n\infty)) \cong H^0(U, \mathscr{E}).$$

Proof If we define $e := 1 \in H^0(X, \mathcal{O}_X(\infty))$, then with the notation of [Liu02], chapter 5, definition 1.24, page 166 we have

$$X_e := \{x \in X \mid \mathcal{O}_X(\infty)_x = e_x \mathcal{O}_{X,x}\} = U.$$

The assumption follows now from [Liu02], chapter 5, lemma 1.25., a) and b), page 166. □

Lemma 2.31
If $B = \bigoplus_{i=0}^{\infty} B_i$ is a graduate algebra then for all $n \in \mathbb{N}$ there exists a canonical isomorphism

$$\operatorname{Proj} B \cong \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} B_{in}\right).$$

If B is a finitely generated B_0-algebra then there exists a number n_0 such that the graduate algebra $\bigoplus_{i=0}^{\infty} B_{in}$ is generated by B_n for all $n \geq n_0$.

Proof Cf. [Liu02], chapter 2, exercise 3.11, page 57 and [Bou98], chapter III, § 1.3, Proposition 3. □

Definition 2.32 ([GD61], Abschnitt 3.3)
Let B be a graduate ring and define $X := \operatorname{Proj} B$. Let $\mathscr{E}$ be an $\mathcal{O}_X$-module and define $\mathscr{E}(n) := \mathscr{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ for all $n \in \mathbb{Z}$. We define by

$$\Gamma_*(\mathscr{E}) := \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathscr{E}(n))$$

the associated, graduated B-module of $\mathscr{E}$.

Proposition 2.33
Let B be a graduated ring, finitely generated as a B_0-algebra. Define $X :=$ $\operatorname{Proj} B$ and let $\mathscr{E}$ be a quasi coherent $\mathcal{O}_X$-module. Then there is a canonical isomorphism

$$\Gamma_*(\mathscr{E})^{\sim} \cong \mathscr{E}.$$

Proof From lemma 2.31 we conclude without loss of generality that B is generated by B_1 as a B_0-algebra. The assumption follows now from [Har77], II, chapter 5, proposition 5.15, page 119. □

Definition 2.34
We define

$$\mathcal{S}_X := \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i\infty)).$$

Lemma 2.35
$\mathcal{S}_X$ is a finitely generated, graduated K-algebra.

Proof From proposition 2.27 we know $H^0(X, \mathcal{O}_X) = K$. For all $i, j \in \mathbb{Z}$ there exists canonical maps

$$\mathcal{O}_X(i\infty) \otimes_{\mathcal{O}_X} \mathcal{O}_X(j\infty) \longrightarrow \mathcal{O}_X((i+j)\infty).$$

They define the structure of a graduated K-algebra on $\mathcal{S}_X$. If $U \subseteq X$ is an affine open subset then $H^0(U, \mathcal{O}_X)$ is a finite generated K-algebra. If $f_1, \ldots, f_n \in H^0(U, \mathcal{O}_X)$ is a set of generators then by proposition 2.30 there is a number $n_0 \in \mathbb{N}$ such that $f_1, \ldots, f_n \in H^0(X, \mathcal{O}_X(n_0\infty))$. In particular this elements generate $\mathcal{S}_X$ as a graduate K-algebra. □

Proposition 2.36
We have $\operatorname{Proj} \mathcal{S}_X \cong X$.

Proof From proposition 2.29 there exits a number m and an embedding $f : X \longrightarrow \mathbb{P}^n_K$ such that $f^*(\mathcal{O}_{\mathbb{P}^n_K}(1)) \cong \mathcal{O}_X(m\infty)$. The embedding f implies the exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n_K} \xrightarrow{f^\sharp} f_*\mathcal{O}_X \longrightarrow 0$$

with $\mathscr{I} := \operatorname{Ker} f^\sharp$. For all $i > 0$ we tensorize this sequence with $\mathcal{O}_{\mathbb{P}^n_K}(i)$. We get with the help of the projection formula (cf. lemma 2.22, 2)

$$f_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} f^*\mathcal{O}_{\mathbb{P}^n_K}(i)) \cong f_*\mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^n_K}} \mathcal{O}_{\mathbb{P}^n_K}(i)$$

the sequence

$$0 \longrightarrow \mathscr{I}(i) \longrightarrow \mathcal{O}_{\mathbb{P}^n_K}(i) \longrightarrow f_*(\mathcal{O}_X(i)) \longrightarrow 0.$$

With the help of

$$f_*(\mathcal{O}_X(i))(\mathbb{P}^n_K) = H^0(X, \mathcal{O}_X(mi\infty))$$

and using global sections we get the sequence

$$0 \longrightarrow H^0(\mathbb{P}^n_K, \mathscr{I}(i)) \longrightarrow H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(i)) \longrightarrow H^0(X, \mathcal{O}_X(mi\infty)) \longrightarrow$$
$$\longrightarrow H^1(\mathbb{P}^n_K, \mathscr{I}(i)) \longrightarrow \cdots.$$

There exists a number $d > 0$ such that $H^1(\mathbb{P}^n_K, \mathscr{I}(i)) = 0$ for all $i \geq d$. In particular

$$H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(i))/H^0(\mathbb{P}^n_K, \mathscr{I}(i)) \cong H^0(X, \mathcal{O}_X(mi\infty))$$

for all $i \geq d$. In addition $H^0(\mathbb{P}^n_K, \mathscr{I}) = 0$ or else $\mathscr{I} = \mathcal{O}_{\mathbb{P}^n_K}$ which is impossible. From proposition 2.27 we conclude $H^0(X, \mathcal{O}_X) = K$. It follows

$$\begin{aligned}\operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i\infty))\right) &\cong \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(mdi\infty))\right)\\ &\cong \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(di))/H^0(\mathbb{P}^n_K, \mathscr{I}(di))\right)\\ &\cong \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(i))/H^0(\mathbb{P}^n_K, \mathscr{I}(i))\right).\end{aligned}$$

From [Har77], part II, proposition 5.13, 5.15, corollary 5.16, page 118 ff, we conclude

$$\begin{aligned}\operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(i))/H^0(\mathbb{P}^n_K, \mathscr{I}(i))\right) &\cong \operatorname{Proj}\left(K[x_0, \ldots, x_n]/\Gamma_*(\mathscr{I})\right)\\ &\cong X.\end{aligned}$$

This proofs the assumption. □

Corollary 2.37
We have for all $j \in \mathbb{Z}$

$$(\mathcal{S}_X[j])^{\sim} \cong \mathcal{O}_X(j\infty).$$

Proof From proposition 2.33 we conclude

$$\Gamma_*(\mathcal{O}_X(j\infty))^{\sim} \cong \mathcal{O}_X(j\infty).$$

We have $\mathcal{O}_X(1) = \mathcal{O}_X(m\infty)$ and this implies

$$\bigoplus_{i=0}^{\infty} H^0(\mathcal{O}_X(j\infty) \otimes \mathcal{O}_X(im\infty)) \cong \bigoplus_{i=0}^{\infty} H^0(\mathcal{O}_X(j + im\infty)).$$

From [Har77], part II, exercise 5.9, page 125 it follows

$$\left(\bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i\infty))[j]\right)^{\sim} \cong \left(\bigoplus_{i=0}^{\infty} H^0(\mathcal{O}_X(j + im\infty))\right)^{\sim}.$$ □

Corollary 2.38
We define $A := H^0(X \setminus \infty, \mathcal{O}_X)$ and $e := 1 \in H^0(X, \mathcal{O}_X(\infty))$. This gives

$$\mathcal{S}_X[e^{-1}]_{(0)} \cong A.$$

Proof In the notation of [Har77], part II, lemma 5.14, page 118, we have $X_e = X \setminus \infty$. From loc. cit. a), b) we conclude that the canonical map

$$\mathcal{S}_X[e^{-1}]_{(0)} \ni \frac{f}{e^i} \longmapsto f \in A$$

is an isomorphism. □

Corollary 2.39
Let $e := 1 \in H^0(X, \mathcal{O}_X(\infty))$. This gives

$$\operatorname{Proj}(\mathcal{S}_X / e\mathcal{S}_X) \cong \operatorname{spec} K(\infty).$$

Proof We have

$$(\mathcal{S}_X)/e(\mathcal{S}_X) \cong \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i\infty))/H^0(X, \mathcal{O}_X((i-1)\infty)).$$

If π_∞ is a uniformizer element of $\mathcal{O}_{X,\infty}$ and if $f \in H^0(X, \mathcal{O}_X(i\infty))$ such that $\infty(f) = -i$ we can define for all $i \geq 0$ the maps

$$\begin{aligned} H^0(X, \mathcal{O}_X(i\infty))/H^0(X, \mathcal{O}_X((i-1)\infty)) &\longrightarrow K(\infty)[T]_{(i)} \\ f &\longmapsto (f\pi_\infty^i \bmod \mathfrak{m}_\infty)T^i. \end{aligned}$$

If $i \gg 0$ the maps are isomorphisms. This implies the assumption. □

2.5 Vector bundles

Definition 2.40
Let S be a scheme. We call a quasi coherent sheaf $\mathscr{E}$ of $\mathcal{O}_S$-modules a vector bundle *on S if for all open affine subsets $U \subseteq S$ the $\mathcal{O}_S(U)$-module $H^0(U, \mathscr{E})$ is projective (cf. [Dri03]).*

Theorem 2.41
Let $\mathscr{E}$ be a quasi-coherent $\mathcal{O}_S$-module. $\mathscr{E}$ is a vector bundle if there exists an open covering of S $\bigcup_{i \in I} U_i = S$ such that $\mathscr{E}|_{U_i}$ is a vector bundle over U_i for all $i \in I$.

Proof Cf. [Dri03] or [RG71], §3, 3.1 *Descent de la projectivité*, page 81 ff. □

Remark 2.42
The conclusion of the theorem is also true for a covering in the sense of the fpqc topology (cf. loc. cite.)

Lemma 2.43
Let S be a scheme and let

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

be an exact sequence of quasi-coherent $\mathcal{O}_S$-modules. If $\mathscr{F}$ and $\mathscr{G}$ are vector bundles then $\mathscr{E}$ is a vector bundle too.

Proof The statement is by theorem 2.41 local in the Zariski topology. Hence let $S = \operatorname{spec} R$ be an affine scheme. Then it follows that the sequence

$$0 \longrightarrow \mathscr{E}(S) \longrightarrow \mathscr{F}(S) \longrightarrow \mathscr{G}(S) \longrightarrow 0$$

is split and $\mathscr{E}(S)$ is a direct summand of the projective R-module $\mathscr{F}(S)$. So it is itself projective. □

2.6 Cohomology and base change of vector bundles on parametrised curves

Let S be a scheme over K. We call $X \times S$ a parametrised curve over S. Let be $s \in S$ and let $k(s)$ be the residue field of the point s. We define X_S respectively X_s to be the fibre product $X \times S$ respectively $X \times \operatorname{spec} k(s)$. From corollary 2.25 we get that with X also X_s is a smooth projective absolutely irreducible curve over $k(s)$ for all points $s \in S$.
If $\mathscr{E}$ is an $\mathcal{O}_{X\times S}$-module then we define $\mathscr{E}_s$ to be the pullback of the sheaf $\mathscr{E}$ on $X \times s$.
The goal of this section is to give a generalisation of the following theorem in the case of a non noetherian base scheme. To gain this goal we need some additional conditions on the vector bundle $\mathscr{E}$.

Theorem 2.44
Let S be a local noetherian scheme. If

$$H^k(X \times s, \mathscr{E}(i\infty)_s) = 0 \quad \forall s \in S$$

then $H^k(X \times S, \mathscr{E}(i\infty)) = 0$ for $k = 0, 1$.

Proof Cf. [Liu02], chapter 5, remark 3.21(c), page 204. □

Lemma 2.45
Let $S = \operatorname{spec} R$ be an affine scheme and let $\mathscr{E}$ be a vector bundle of rank d on $X \times S$. There exists a noetherian subring $R_0 \subseteq R$ respectively an affine noetherian scheme $S_0 = \operatorname{spec} R_0$ and a vector bundle $\mathscr{E}_0$ on $X \times S_0$ such that

$$\mathscr{E}_0 \times_{S_0} S \cong \mathscr{E}.$$

Proof As $X \times S$ is separated and quasi-compact it exists a finite open affine covering of $X = \cup_{i=1}^n U_i \times V_i$ such that

1) $U_i = \operatorname{spec} A_i$

2) $V_i = \operatorname{spec} R_{f_i}$ for $f_i \in R$ not a zero-divisor

3) $\mathscr{E}|_{U_i \times V_i}$ is a free $A_i \otimes R_{f_i}$-module of $\operatorname{rk} d$

We conlude that

$$U_i \times V_i \cap U_j \times V_j = \operatorname{spec}(A_{ij} \otimes R_{f_i f_j})$$

for $A_{ij} = U_i \cap U_j$. The datas of the vector bundle are given by the covering and the matrices

$$\alpha^{(ij)} \in \mathrm{GL}_d(A_{ij} \otimes R_{f_i f_j})$$

for $1 \leq i, j \leq n$. Let

$$\mathcal{R} := \{\alpha_{kl}^{(ij)} \text{ for } 1 \leq i, j, \leq n \text{ and } 1 \leq k, l \leq d\}$$

and let $f := f_1 \cdot \ldots \cdot f_n$. Then it exits a number $N \in \mathbb{N}$ such that $f^N x \in R$ for all $x \in \mathcal{R}$. As $\operatorname{spec} R = \cup_{i=1}^n \operatorname{spec} R_{f_i}$ is a covering there exists elements $a_1, \ldots, a_n$ such that $1 = \sum_{i=1}^n a_i f_i$.
If we define

$$\tilde{\mathcal{R}} := \{f^N x \mid x \in \mathcal{R}\} \cup \{a_1, \ldots, a_n\} \cup \{f_1, \ldots, f_n\}$$

then

$$R_0 := \mathbb{F}_q[\tilde{\mathcal{R}}] \subseteq R$$

is a finitely generated $\mathbb{F}_q$-subalgebra of R. In particular we have

$$\alpha^{(ij)} \in \mathrm{GL}_d(A_{ij} \otimes R_{0 f_i f_j})$$

and a covering $X \times \operatorname{spec} R_0 = \cup_{i=1}^d U_i \times \operatorname{spec} R_{0 f_i}$. Thus we can define a vector bundel $\mathscr{E}_0$ on $X \times S_0$ with the desired property. □

Lemma 2.46
Let $S = \operatorname{spec} R$ be an affine scheme. Let $U, V \subseteq X$ be open affine subsets of X covering X. Then the Čech complex

$$0 \longrightarrow \mathscr{E}(U \times S) \times \mathscr{E}(V \times S) \xrightarrow{\psi} \mathscr{E}((U \cap V) \times S) \longrightarrow 0$$

calculates the cohomology of the vector bundle $\mathscr{E}$.

Proof Cf. [Liu02], chapter 5, theorem 2.19, page 186. □

Lemma 2.47
Let R be a ring and let M be a projective, finitely generated R-module.

1) Let $\mathfrak{p}$ be a prime ideal of R, then we have

$$M_{\mathfrak{p}} = 0 \quad \Longleftrightarrow \quad M \otimes_R k(\mathfrak{p}) = 0.$$

2) Let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{p} \subseteq \mathfrak{m}$. Then we have

$$M \otimes_R k(\mathfrak{m}) = 0 \quad \Longleftrightarrow \quad M \otimes_R k(\mathfrak{p}) = 0.$$

Proof 1) As M is a projective, finitely generated R-Module M is locally free that is it exits a number $n \in \mathbb{N}$ such that $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$. We conclude

$$M \otimes_R k(\mathfrak{p}) \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} k(\mathfrak{p}) \cong R_{\mathfrak{p}}^n \otimes_{R_{\mathfrak{p}}} k(\mathfrak{p}) \cong k(\mathfrak{p})^n.$$

2) According to part 1) it is sufficient to show that

$$M_{\mathfrak{m}} = 0 \quad \Longleftrightarrow \quad M_{\mathfrak{p}} = 0.$$

It is

$$(M_{\mathfrak{m}})_{\mathfrak{p}} \cong (R_{\mathfrak{m}}^n)_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n \cong M_{\mathfrak{p}}$$

and we get the assumption. □

Proposition 2.48
In the setting of Lemma 2.45 let $H^1(X \times S_0, \mathscr{E}_0)$ be a projective finitely generated R_0-module. If $H^1(X \times s, \mathscr{E}_s) = 0$ for all $s \in S$ then $H^1(X \times S, \mathscr{E}) = 0$.

Proof With the help of the canonical map $S \longrightarrow S_0$ we regard the points of S as points of S_0. We get

$$\mathscr{E}_s = (\mathscr{E}_0)_s$$

for all points $s \in S$. As S_0 is noetherian if follows form [Liu02], chapter 5, remark 3.21, (b), (c), page 204,

$$H^1(X \times s, (\mathscr{E}_0)_s) = 0 \quad \Longrightarrow \quad H^1(X \times S_0, \mathscr{E}_0) \otimes_{R_0} k(s) = 0$$

for all $s \in S$. As S is dense in S_0 it follows from lemma 2.47, 1) and 2) that $H^1(X \times S_0, \mathscr{E}_0) = 0$. As in Lemma 2.46 the Čech complex

$$0 \longrightarrow \mathscr{E}_0(U \times S_0) \times \mathscr{E}_0(V \times S_0) \xrightarrow{\psi_0} \mathscr{E}_0((U \cap V) \times S_0) \longrightarrow 0$$

calculates the cohomology of $\mathscr{E}_0$. This implies the surjectivity of ψ_0. If we tensorize the sequence with $\cdot \otimes_{R_0} R$ we gain the Čech complex which calculates the cohomology of $\mathscr{E}$. Because tensorizing is right exact the map $\psi_0 \otimes_{R_0} R$ stays surjectiv. From this we get the assumption. □

Proposition 2.49
In the setting of lemma 2.45 let $H^0(X \times S_0, \mathscr{E}_0)$ be a projective, finitely generated R_0 module. In addition let $H^1(X \times S_0, \mathscr{E}_0)$ be a flat R_0 module. If $H^0(X \times s, \mathscr{E}_s) = 0$ for all $s \in S$ then it follows $H^0(X \times S, \mathscr{E}) = 0$.

Proof From the Čech complex

$$0 \longrightarrow \mathscr{E}_0(U \times S_0) \times \mathscr{E}_0(V \times S_0) \xrightarrow{\psi_0} \mathscr{E}_0((U \cap V) \times S_0) \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow \operatorname{Im} \psi_0 \longrightarrow \mathscr{E}_0((U \cap V) \times S_0) \longrightarrow H^1(X \times S_0, \mathscr{E}_0) \longrightarrow 0.$$

As $H^1(X \times S_0, \mathscr{E}_0)$ and $\mathscr{E}((U \cap V) \times S_0)$ (cf. Lemma 2.6) are flat R_0-modules also $\operatorname{Im} \psi_0$ is a flat R_0-module (cf. Lemma 2.12). We tensorize the sequence

$$0 \longrightarrow H^0(X \times S_0, \mathscr{E}_0) \longrightarrow \mathscr{E}_0(U \times S_0) \times \mathscr{E}_0(V \times S_0) \longrightarrow \operatorname{Im} \psi_0 \longrightarrow 0$$

with $\cdot \otimes_{R_0} R$. Because $\operatorname{Im} \psi_0$ is a flat R_0-module we conclude

$$H^0(X \times S_0, \mathscr{E}_0) \otimes_{R_0} R \cong H^0(X \times S, \mathscr{E}).$$

On the lines of the proof of proposition 2.48 we conclude $H^0(X \times S_0, \mathscr{E}_0) = 0$ from $H^0(X \times s, (\mathscr{E}_0)_s) = 0$ for all points $s \in S$. This proofs the assumption.

□

Corollary 2.50
In the setting of lemma 2.45 let $H^1(X \times S, \mathscr{E})$ be a flat R-module. Let $S' = \operatorname{spec} R'$ be an affine scheme over S. We have

$$H^0(X \times S, \mathscr{E}) \otimes_R R' \cong H^0(X \times S', \mathscr{E}_{S'})$$

and

$$H^1(X \times S, \mathscr{E}) \otimes_R R' \cong H^1(X \times S', \mathscr{E}_{S'}).$$

Proof From lemma 2.46 we know that the sequences

$$0 \longrightarrow H^0(X \times S, \mathscr{E}) \longrightarrow \mathscr{E}(U \times S) \times \mathscr{E}(V \times S) \xrightarrow{\psi} \operatorname{Im}\psi \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im}\psi \longrightarrow \mathscr{E}((U \cap V) \times S) \longrightarrow H^1(X \times S, \mathscr{E}) \longrightarrow 0.$$

calculates the cohomology of $\mathscr{E}$. If $H^1(X \times S, \mathscr{E})$ is a flat R-Module then the sequences remain exact under tensorizing with $\otimes_R R'$ and calculate the cohomology of $\mathscr{E}_{S'}$. □

2.7 Twists of vector bundles on parametrized curves

Definition 2.51
Let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$-module. For all $n \in \mathbb{Z}$ let

$$\mathscr{E}(n\infty) := \mathscr{E} \otimes_{\mathcal{O}_{X\times S}} \operatorname{pr}_X^* \mathcal{O}_X(n\infty)$$

be the twist with the line bundle $\operatorname{pr}_X^* \mathcal{O}_X(n\infty)$. In particular

$$\mathcal{O}_{X\times S}(n\infty) \cong \operatorname{pr}_X^* \mathcal{O}_X(n\infty).$$

Remark 2.52
For all $n, m \in \mathbb{Z}$ we have

$$\mathcal{O}_{X\times S}(n\infty) \otimes_{\mathcal{O}_{X\times S}} \mathcal{O}_{X\times S}(m\infty) \cong \mathcal{O}_{X\times S}((n+m)\infty)$$

and

$$\mathscr{E}(n\infty) \otimes_{\mathcal{O}_{X\times S}} \mathcal{O}_{X\times S}(m\infty) \cong \mathscr{E}((n+m)\infty).$$

Proposition 2.53
There exists a number $m \in \mathbb{N}$ such that the line bundle $\mathcal{O}_{X\times S}(m\infty)$ is very ample.

Proof From proposition 2.29 we know the existence of a number $m \in \mathbb{N}$ and a closed embedding $f : X \longrightarrow \mathbb{P}^n_K$ such that $\mathcal{O}_X(m\infty) = f^*\mathcal{O}_{\mathbb{P}^n_K}(1)$. From [Har77], part II, exercise 3.11, page 92, we conclude that

$$f \times \mathrm{id}_S : X \times S \longrightarrow \mathbb{P}^n_K \times S = \mathbb{P}^n_S$$

is a closed embedding. We get

$$(f\times\mathrm{id}_S)^*\mathcal{O}_{\mathbb{P}^n_S}(1) = (f\times\mathrm{id}_S)^* \operatorname{pr}^*_{\mathbb{P}^n_K} \mathcal{O}_{\mathbb{P}^n_K}(1) = \operatorname{pr}_X^* f^*\mathcal{O}_{\mathbb{P}^n_K}(1) = \operatorname{pr}_X^* \mathcal{O}_X(m\infty).$$

Lemma 2.54
Let $S = \operatorname{spec} R$ be an affine scheme and let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$ vector bundle of finite rank. Then there exists numbers $k, r \in \mathbb{Z}$ and an $\mathcal{O}_{X\times S}$ vector bundle $\mathscr{F}$ of finite rank such that there exists an exact sequence of the form

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_{X\times S}(k\infty)^{\oplus r} \longrightarrow \mathscr{E} \longrightarrow 0.$$

Proof From proposition 2.53 we conclude that for an appropriate $m \in \mathbb{Z}$ the vector bundle $\mathcal{O}_{X\times S}(m\infty)$ is very ample. By [Liu02], chapter 5, theorem 1.27, page 167, there exists a number $n \in \mathbb{Z}$ such that $\mathscr{E}(nm\infty)$ is generated by global sections. In particular for an appropriate $r \in \mathbb{N}$ exists a surjective map

$$\mathcal{O}_{X\times S}^{\oplus r} \longrightarrow \mathscr{E}(nm\infty) \longrightarrow 0.$$

Tensorizing of the sequence by $\mathcal{O}_{X\times S}(-nm\infty)$ gives

$$\mathcal{O}_{X\times S}((-nm)\infty)^{\oplus r} \longrightarrow \mathscr{E} \longrightarrow 0.$$

By lemma 2.43 the kernel of this map, called $\mathscr{F}$, is a vector bundle of finite rank. □

Lemma 2.55
Let $S = \operatorname{spec} R$ be an affine scheme and let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$ vector bundle of finite rank. Then there exists numbers $m, n \in \mathbb{Z}$ such that $H^0(X \times S, \mathscr{E}(n\infty)) \neq 0$ and such that $H^0(X \times S, \mathscr{E}(m\infty)) = 0$.

Proof

a) As for an appropriate $k \in \mathbb{N}$ the line bundle $\mathcal{O}_{X\times S}(k\infty)$ is very ample there exists a number $n' \in \mathbb{N}$ such that $\mathscr{E}(n'k\infty)$ is generated by global sections. In particular for $n := n'k$ we get $H^0(X \times S, \mathscr{E}(n\infty)) \neq 0$.

b) For $k < 0$ there are no global sections of $\mathcal{O}_X(k\infty)$. As R/K is a flat base change we have for $k < 0$

$$H^0(X \times S, \mathcal{O}_{X\times S}(k\infty)) \cong H^0(X, \mathcal{O}_X(k\infty)) \otimes_K R = 0.$$

c) Let $\mathscr{E}^\vee$ be the dual vector bundle. From lemma 2.54 we conclude the existence of numbers $n^\vee, r \in \mathbb{N}$ such that there is an exact sequence of the form

$$\mathcal{O}_{X\times S}(-n^\vee\infty)^{\oplus r} \longrightarrow \mathscr{E}^\vee \longrightarrow 0.$$

Dualizing leads to

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_{X\times S}(n^\vee\infty)^{\oplus r}.$$

We choose $m < -n^\vee$. Then if we tensorize with $\mathcal{O}_{X\times S}(m\infty)$ we get

$$0 \longrightarrow \mathscr{E}(m\infty) \longrightarrow \mathcal{O}_{X\times S}((n^\vee + m)\infty)^{\oplus r}.$$

As because of the choice of m the bundle $\mathcal{O}_{X\times S}((n^\vee+m)\infty)$ has no global sections, also $\mathscr{E}(m\infty)$ has no global sections. □

Let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$ vector bundle of rank $d/\deg(\infty)$ and define for all $i \in \mathbb{Z}$

$$\mathscr{E}_i := \mathscr{E}(i\infty).$$

Below we show that for all $i \in \mathbb{Z}$ the $\mathcal{O}_S$-modules $\mathrm{pr}_{S*}\,\mathscr{E}_i/\mathscr{E}_{i-1}$ are vector bundles of rank d.

Lemma 2.56
For all $i \in \mathbb{Z}$ we have

$$\mathscr{E}_i/\mathscr{E}_{i-1} \cong \mathscr{E} \otimes_{\mathcal{O}_{X\times S}} \mathcal{O}_{X\times S}(i\infty)/\mathcal{O}_{X\times S}((i-1)\infty).$$

Proof We tensorize the exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathcal{O}_{X\times S}((i-1)\infty) \longrightarrow \mathcal{O}_{X\times S}(i\infty) \longrightarrow \\ \qquad \longrightarrow \mathcal{O}_{X\times S}(i\infty)/\mathcal{O}_{X\times S}((i-1)\infty) \longrightarrow 0 \end{array}$$

with $\mathscr{E}$. This gives the assumption. □

Lemma 2.57
For all $i \in \mathbb{Z}$ we have

$$\mathcal{O}_{X\times S}(i\infty)/\mathcal{O}_{X\times S}((i-1)\infty) \cong \mathrm{pr}_X^*\,\mathcal{O}_X(i\infty)/\mathcal{O}_X((i-1)\infty).$$

Proof The functor pr_X^* is an exact functor because $\mathcal{O}_{X\times S}$ is a flat $\mathcal{O}_X$-module. We apply this functor to the sequence

$$0 \longrightarrow \mathcal{O}_X((i-1)\infty) \longrightarrow \mathcal{O}_X(i\infty) \longrightarrow \mathcal{O}_X(i\infty)/\mathcal{O}_X((i-1)\infty) \longrightarrow 0.$$

This proofs the assumption. □

Remark 2.58
The ideal sheaf $\mathcal{O}_X(-\infty)$ corresponds to the closed subscheme $\operatorname{spec} k(\infty) \hookrightarrow X$. As $\mathcal{O}_X(-\infty)$ is a flat $\mathcal{O}_X$-module we conclude that the ideal sheaf $\mathcal{O}_{X\times S}$ corresponds to the closed subscheme $k(\infty) \times S \hookrightarrow X \times S$.

Lemma 2.59
Let $\iota : \operatorname{spec} k(\infty) \longrightarrow X$ *be the canonical embedding. For all numbers* $i \in \mathbb{Z}$ *we have*

$$\mathcal{O}_X(i\infty)/\mathcal{O}_X((i-1)\infty) \cong \iota_* k(\infty).$$

Proof The sheaf of ideals $\mathcal{O}_X(-\infty)$ annulates for all $i \in \mathbb{Z}$ the $\mathcal{O}_X$-module $\mathcal{O}_X(i\infty)/\mathcal{O}_X((i-1)\infty)$. In addition we have $\iota^*\mathcal{O}_X(i\infty)/\mathcal{O}_X((i-1)\infty) \cong k(\infty)$. The assumption follows now from lemma 2.21, 3). □

Corollary 2.60
Let be $\iota_S : \operatorname{spec} k(\infty) \times S \longrightarrow X \times S$*. For all* $i \in \mathbb{Z}$ *we have*

$$\mathcal{O}_{X\times S}(i\infty)/\mathcal{O}_{X\times S}((i-1)\infty) \cong \iota_{S*}\mathcal{O}_{k(\infty)\times S}.$$

Proof As with $S \longrightarrow \operatorname{spec} K$ the map $X \times S \longrightarrow X$ is flat. The assumption follows now from lemma 2.22, 1). □

Lemma 2.61
Let $\iota_S : \operatorname{spec} k(\infty) \times S \longrightarrow X \times S$*. For all* $i \in \mathbb{Z}$ *we have*

$$\mathcal{E}_i/\mathcal{E}_{i-1} \cong \iota_{S*}\iota_S^*\mathcal{E}_i/\mathcal{E}_{i-1}.$$

Proof From remark 2.58 we conclude that the sheaf of ideals corresponding to the closed subscheme $\operatorname{spec} k(\infty) \times S$ is $\mathcal{O}_{X\times S}(-\infty)$. Then we get

$$\mathcal{E}_{i-1} \cong \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} \mathcal{O}_{X\times S}(-\infty) \cong \mathcal{O}_{X\times S}(-\infty)\mathcal{E}_i.$$

In particular the sheaf of ideals $\mathcal{O}_{X\times S}(-\infty)$ annulates $\mathcal{E}_i/\mathcal{E}_{i-1}$. The assumption follows now from lemma 2.21, 3). □

Lemma 2.62
For all $i \in \mathbb{Z}$ *we have*

$$\iota_S^*\mathcal{E}_i/\mathcal{E}_{i-1} \cong \iota_S^*\mathcal{E}.$$

Proof From the commuting diagram

$$\begin{array}{ccc}
\operatorname{spec} k(\infty) \times S & \xrightarrow{\iota_S} & X \times S \\
{\scriptstyle \operatorname{pr}_{\operatorname{spec} k(\infty)}}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{pr}_X} \\
\operatorname{spec} k(\infty) & \xrightarrow{\iota} & X
\end{array}$$

it follows

$$\begin{aligned}\iota_S^*\left(\mathscr{E}_i/\mathscr{E}_{i-1}\right) &\cong \iota_S^*\left(\mathscr{E}\otimes_{\mathcal{O}_{X\times S}}\operatorname{pr}_X^*\iota_*k(\infty)\right) \qquad \text{(cf. lemma 2.56, 2.57, 2.59)}\\ &\cong \iota_S^*\mathscr{E}\otimes_{\mathcal{O}_{\operatorname{spec}k(\infty)\times S}}\iota_S^*\operatorname{pr}_X^*\iota_*k(\infty)\\ &\cong \iota_S^*\mathscr{E}\otimes_{\mathcal{O}_{\operatorname{spec}k(\infty)\times S}}\operatorname{pr}_{\operatorname{spec}k(\infty)}^*\iota^*\iota_*k(\infty)\\ &\cong \iota_S^*\mathscr{E}\otimes_{\mathcal{O}_{\operatorname{spec}k(\infty)\times S}}\operatorname{pr}_{\operatorname{spec}k(\infty)}^*k(\infty) \qquad \text{(cf. lemma 2.21, 1)}\\ &\cong \iota_S^*\mathscr{E}.\end{aligned}$$

Corollary 2.63
For all $i\in\mathbb{Z}$ the sheaf $\operatorname{pr}_{S*}\mathscr{E}_i/\mathscr{E}_{i-1}$ is an $\mathcal{O}_S$-vector bundle of rank d.

Proof From lemma 2.61 we conclude

$$\operatorname{pr}_{S*}\mathscr{E}_i/\mathscr{E}_{i-1}\cong\operatorname{pr}_{S*}\iota_{S*}\iota_S^*\mathscr{E}_i/\mathscr{E}_{i-1}.$$

and it is sufficient to show the assumption for $\iota_S^*\left(\mathscr{E}_i/\mathscr{E}_{i-1}\right)$. As pullback of the vector bundle $\mathscr{E}$ the sheaf $\iota_S^*\left(\mathscr{E}_i/\mathscr{E}_{i-1}\right)$ is an $\mathcal{O}_{\operatorname{spec}k(\infty)\times S}$-vector bundle of rank $\frac{d}{\deg(\infty)}$. The finite field extension $k(\infty)/k$ has degree $\deg(\infty)$. From this we conclude that $\operatorname{pr}_{S*}\iota_S^*\left(\mathscr{E}_i/\mathscr{E}_{i-1}\right)$ is an $\mathcal{O}_S$-vector bundle of rank d. □

Lemma 2.64
Let $S=\operatorname{spec}R$ be an affine scheme and let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$ vector bundle. Then $H^1(X\times S,\mathscr{E}_i/\mathscr{E}_{i-1})=0$ for all $i\in\mathbb{Z}$.

Proof As S is an affine scheme the support of $\mathscr{E}_i/\mathscr{E}_{i-1}$ is affine too (cf. lemma 2.61). The assumption follows (cf. [Liu02], chapter 5, exercise 2.3 (b), page 191 and theorem 2.18, page 186). □

Corollary 2.65
Let $S=\operatorname{spec}R$ be an affine scheme and let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$-vector bundle. Then there exists a number $n_0\in\mathbb{Z}$ such that $H^0(X\times S,\mathscr{E}(i\infty))=0$ for $i\leq n_0$ and $H^0(X\times S,\mathscr{E}(i\infty))\neq 0$ for $i>n_0$.

Proof From the exact sequence

$$0\longrightarrow\mathscr{E}_i\longrightarrow\mathscr{E}_{i+1}\longrightarrow\mathscr{E}_{i+1}/\mathscr{E}_i\longrightarrow 0$$

we derive the long exact sequence of cohomology

$$\begin{aligned}0\longrightarrow H^0(X\times S,\mathscr{E}_i)\longrightarrow H^0(X\times S,\mathscr{E}_{i+1})\longrightarrow H^0(X\times S,\mathscr{E}_i/\mathscr{E}_{i+1})\longrightarrow\\ \longrightarrow H^1(X\times S,\mathscr{E}_i)\longrightarrow H^1(X\times S,\mathscr{E}_{i+1})\longrightarrow 0.\end{aligned}$$

As the first map is injective we conclude the assumption from lemma 2.55. □

Proposition 2.66
Let $S = \operatorname{spec} R$ be an affine scheme and let $\mathscr{E}$ be a vector bundle on $X \times S$. We set $U := X \setminus \{\infty\}$. For all $n \in \mathbb{Z}$ let

$$H^0(X \times S, \mathscr{E}(n\infty)) \longrightarrow H^0(U \times S, \mathscr{E}(n\infty)) = H^0(U \times S, \mathscr{E})$$

be the canonical maps and for all $n, m \in \mathbb{Z}$, $n \leq m$, let

$$H^0(X \times S, \mathscr{E}(n\infty)) \longrightarrow H^0(X \times S, \mathscr{E}(m\infty))$$

be the canonical inclusions. Then we have

$$\varinjlim_n H^0(X \times S, \mathscr{E}(n\infty)) \cong H^0(U \times S, \mathscr{E}).$$

Proof The statement of the proposition is compatible with flat base change on S. Hence without loss of generality we can assume $R = K$. The assumption now follows from proposition 2.30. □

Proposition 2.67
Let $S = \operatorname{spec} R$ be an affine scheme and let $\mathscr{E}$ be a vector bundle on $X \times S$. We set $U := X \setminus \{\infty\}$. Then we have

$$H^0(X \times S, \varinjlim_n \mathscr{E}_n)) = \varinjlim_n H^0(X \times S, \mathscr{E}_n) = H^0(U \times S, \mathscr{E}).$$

Proof Let $\mathscr{F} := \varinjlim_n \mathscr{E}_n$. As with X and S also $X \times S$ is separated and quasi compact. In particular there exits a finite affine open covering $X = \cup_{i=1}^k U_i$. Thus we can calculate the cohomology of the quasi coherent $\mathcal{O}_{X \times S}$-module $\mathscr{F}$ from the Čech complex. We get the exact sequence

$$0 \longrightarrow H^0(X \times S, \mathscr{F}) \longrightarrow \prod_{i=1}^k H^0(U_i, \mathscr{F}) \longrightarrow \prod_{i,j=1}^k H^0(U_{ij}, \mathscr{F}).$$

Here wet set $U_{ij} := U_i \cap U_j$. We show below that the canonical map

$$u : \varinjlim_n H^0(X \times S, \mathscr{E}_n) \longrightarrow H^0(X \times S, \mathscr{F})$$

is an isomorphism.
For the injectivity let $(s_n) \in \varinjlim_n H^0(X \times S, \mathscr{E}_n)$ be such that $u((s_n)) = 0$. Then for every $1 \leq i \leq k$ there exists a number n_i such that $s_n|_{U_i} = 0$ for all $n \geq n_i$. We set $n' := \max_i n_i$. Then $s_n|_{U_i} = 0$ for all $1 \leq i \leq k$ and $n \geq n'$ and we get $(s_n) = 0$.

For the surjectivity let $s \in H^0(X \times S, \mathscr{F})$. Then there exits for all $1 \leq i \leq k$ elements $s_i \in H^0(U_i, \mathscr{F})$ such that $s|_{U_i} = s_i$ and such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all $1 \leq i, j \leq k$.
For all $1 \leq i \leq k$ there exists n_i such that $s_i \in H^0(U_i, \mathscr{E}_n)$ for all $n \geq n_i$. We set $n'' := \max_i n_i$. Then $s_i \in H^0(U_i, \mathscr{E}_n)$ for $1 \leq i \leq k$ and $n \geq n''$.
For all $1 \leq i, j \leq k$ there exits numbers $m_{ij} \geq n''$ such that

$$s_i|_{U_{ij}} = s_j|_{U_{ij}} \in H^0(U_i, \mathscr{E}_n)$$

for all $n \geq m_{ij}$. We set $m' := \max_{i,j} m_{ij}$. Then there exists an element $s \in H^0(X \times S, \mathscr{E}_n)$ such that $s|_{U_i} = s_i$ for all $1 \leq i \leq k$ and all $n \geq m'$. This shows the assumption. □

Remark 2.68
Proposition 2.67 is valid for all quasi compact separated schemes and all inductive filtrated systems of quasi-coherent modules. A general version of the proposition is [GD66], Théoréme (8.5.2).

2.8 Construction of vector bundles on $X \times S$

Proposition 2.69
We can construct a vector bundle of rank d on $X \times S$ by the following data:

- $\mathscr{M}$: *Vector bundle of rank d on* $\operatorname{spec} A \times S$
- $\mathscr{E}_\infty$: *Vector bundle of rank d on* $\operatorname{spec} \mathcal{O}_{X,\infty} \times S$
- *Isomorphism:* $\mathscr{M} \otimes_A F \cong \mathscr{E}_\infty \otimes_{\mathcal{O}_{X,\infty}} F$

Let $P \neq \infty$ be a closed point of X. From proposition 2.26 we can conclude that $U := X \setminus \{\infty\} = \operatorname{spec} A$ and $V := X \setminus \{P\} = \operatorname{spec} B$ form an open affine covering of $X = U \cup V$.

Lemma 2.70
In the above notation we have:

1) $\operatorname{spec} \mathcal{O}_{X,\infty} \times_X U \cong \operatorname{spec} F$

2) $\operatorname{spec} \mathcal{O}_{X,\infty} \times_X \operatorname{spec} \mathcal{O}_{X,\infty} \cong \operatorname{spec} \mathcal{O}_{X,\infty}$

3) $\operatorname{spec} A \times_X \operatorname{spec} A \cong \operatorname{spec} A$

Proof

1) We can describe the fiber product $\operatorname{spec} \mathcal{O}_{X,\infty} \times_X U$ by the covering $U = \operatorname{spec} A$ and $V = \operatorname{spec} B$ of X that is:

$$\begin{array}{ccc} F \otimes_A A & \cdots & \mathcal{O}_{X,\infty} \otimes_B F \\ \| & & \| \\ F & = & F \end{array}$$

This implies 1).

2) As in 1) the covering of X by U and V implies

$$\begin{array}{ccc} F \otimes_A F & \cdots & \mathcal{O}_{X,\infty} \otimes_B \mathcal{O}_{X,\infty} \\ \| & & \| \\ F = (\mathcal{O}_{X,\infty})_\eta & \longleftarrow & \mathcal{O}_{X,\infty} \end{array}$$

and

$$\begin{array}{ccc} A \otimes_A A & \cdots & F \otimes_B F \\ \| & & \| \\ A & \longrightarrow & F = A_\eta \end{array}$$

this shows 2) and 3). □

Proof (of the proposition) By part 2) of the lemma 2.70 we know that $\operatorname{spec} F$ is the fiber product of $\operatorname{spec} A$ and $\operatorname{spec} \mathcal{O}_{X,\infty}$ over X. Base change implies the following diagram:

$$\begin{array}{ccc} \operatorname{spec} F \times S & \longrightarrow & \operatorname{spec} A \times S \\ \downarrow & & \downarrow \text{open (thus flat)} \\ \operatorname{spec} \mathcal{O}_{X,\infty} \times S & \underset{\text{flat}}{\longrightarrow} & X \times S \end{array}$$

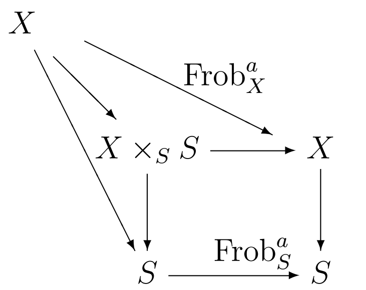
Furthermore by part 2) and part 3) of lemma 2.70 we can conclude that the condition of the faithfully flat decent are fullfilled (cf. [Wat79] 17.2). This proofs the assumption. □

2.9 Frobenius twist of sheafs

Let S be an arbitrary scheme over the finite field $\mathbb{F}_q$. We define the *arithmetic Frobenius endomorphism* $\operatorname{Frob}_S^a : S \longrightarrow S$ as follows:

$$S \xrightarrow{\text{id}} S \text{ and } \mathcal{O}_S(U) \longrightarrow \mathcal{O}_S(U) \quad \text{for all } U \subset X \text{ open}$$
$$f \longmapsto f^q.$$

Let X be a scheme over S. The commutative diagram



defines the pullback $\mathrm{Frob}_S^* X := X \times_S S$ and a morphism, the so called *geometric Frobenius morphism* $\mathrm{Frob}^g : X \longrightarrow \mathrm{Frob}_S^* X$ of schemes over S. In contrast to the geometric Frobenius endomorphism the arithmetic Frobenius endomorphism is in general not a S morphism (cf. [Har77], part IV, chapter 2, page 301 f).

Definition 2.71

1) Let R be a $\mathbb{F}_q$-algebra and let M, N be R-modules. We call an additive map $f : M \longrightarrow N$ q-linear *if for all $r \in R$ and $m \in M$*

$$f(rm) = r^q f(m).$$

2) Let S, T be schemes over $\mathbb{F}_q$. Let $\mathscr{E}$ be a sheaf of modules on $T \times S$. We define the Frobenius twist *of $\mathscr{E}$ (over S) by*

$$^{\tau}\mathscr{E} := (\mathrm{id}_T \times \mathrm{Frob}_S^a)^* \mathscr{E}.$$

3) Let $\mathscr{E}$, $\mathscr{F}$ be $\mathcal{O}_{T\times S}$-modules.
An additive map $\mathscr{E} \longrightarrow \mathscr{F}$ is called q-linear in $\mathcal{O}_S$, linear in $\mathcal{O}_T$ if $\mathrm{pr}_{S*}\mathscr{E} \longrightarrow \mathrm{pr}_{S*}\mathscr{F}$ is an $\mathcal{O}_S$-q-linear map and $\mathrm{pr}_{T*}\mathscr{E} \longrightarrow \mathrm{pr}_{T*}\mathscr{F}$ is an $\mathcal{O}_T$-linear map. We will simply call this maps to be $\mathcal{O}_S$-q-linear.

Remark 2.72

There exists a canonical q-linear map $\mathscr{E} \longrightarrow {}^{\tau}\mathscr{E}$. If $U \subseteq T \times S$ is an open affine subset and if $s \in \mathscr{E}(U)$ then we define

$$\begin{array}{ccc} \mathscr{E}(U) & \longrightarrow & \mathcal{O}_{T\times S}(U) \otimes_{\mathcal{O}_{T\times S}(U)} \mathscr{E}(U) \\ s & \longmapsto & 1 \otimes s. \end{array}$$

Lemma 2.73

Let $S = \operatorname{spec} K$ be a field. Then the canonical map $\mathscr{E} \longrightarrow {}^{\tau}\mathscr{E}$ is injective.

Proof Let $U = \operatorname{spec} R \otimes K \subseteq T \times S$ be an affine open subset. As K is a field the base change Frob : $K \longrightarrow K$ is flat and we conclude ${}^{\tau}\mathscr{E}(U) \cong K \otimes_K \mathscr{E}(U)$. Tensorizing of the injective map Frob : $K \longrightarrow K$ by $\mathscr{E}(U)$ shows the assumption. □

Remark 2.74
An additive map $\mathscr{E} \longrightarrow \mathscr{F}$ of $\mathcal{O}_{T\times S}$-modules is $\mathcal{O}_{T\times S}$-linear iff $\mathrm{pr}_{S*}\mathscr{E} \longrightarrow \mathrm{pr}_{S*}\mathscr{F}$ is a $\mathcal{O}_S$-linear map and $\mathrm{pr}_{T*}\mathscr{E} \longrightarrow \mathrm{pr}_{T*}\mathscr{F}$ is an $\mathcal{O}_T$-linear map.

Proposition 2.75
Let $\mathscr{E}$, $\mathscr{F}$ be sheaf of modules on $T \times S$ and let $f : \mathscr{E} \longrightarrow \mathscr{F}$ be an $\mathcal{O}_S$-q-linear map. Then there exists one and only one $\mathcal{O}_{T\times S}$ linear map ${}^\tau f : {}^\tau\mathscr{E} \longrightarrow \mathscr{F}$ such that the diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\;f\;} & \mathscr{F} \\ & \searrow \qquad \nearrow {\scriptstyle {}^\tau f} & \\ & {}^\tau\mathscr{E} & \end{array}$$

commutes.

Proof

1) If $U \subseteq T \times S$ is an affine open subset we define

$$\begin{array}{ccc} \mathcal{O}_{T\times S}(U) \otimes_{\mathcal{O}_{T\times S}(U)} \mathscr{E}(U) & \longrightarrow & \mathscr{F}(U) \\ x \otimes s & \longmapsto & x f(s). \end{array}$$

   This map is additive and linear in $\mathcal{O}_S$ and $\mathcal{O}_T$ so it is $\mathcal{O}_{T\times S}$-linear.

2) The map ${}^\tau f$ is determined by the image of $\mathscr{E}(U)$ in ${}^\tau\mathscr{E}(U)$ and the commutative diagram. As ${}^\tau\mathscr{E}(U)$ is generated by this image the map determined.

$\square$

3 Drinfeld modules and associated vector bundles

3.1 Drinfeld modules

Let $X/\mathbb{F}_q$ be a smooth, absolute irreducible curve, $F := F(X)$ the function field of X and $\infty \in X$ a closed point. Let $\mathcal{O}_{X,P}$ be the local ring of the point $P \in X$. The corresponding maximal ideal is called $\mathfrak{m}_{X,P}$ and $k(P) := \mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$ is the residue field. We define $\deg(P) := [k(P) : \mathbb{F}_q]$ to be the degree of the residue field over $\mathbb{F}_q$. As the curve X is smooth all local rings $\mathcal{O}_{X,P}$ are discrete valuation rings. We call the corresponding valuations an their canonical continuation on the function field F v_P or if its is clear from the context simply P. The ring

$$A := \Gamma(X \setminus \infty, \mathcal{O}_X) = \{x \in F \mid v_P(x) \geq 0 \; \forall P \in X \setminus \infty\}$$

is a Dedekind ring. For all $x \in F$ we have the product formula

$$\sum_{P \in X} \deg(P) v_P(x) = 0.$$

This implies that $\infty(a) \leq 0$ for all $a \in A$. Because of this fact we define $\deg a := -\infty(a)$ for all elements $a \in F$.
Let $S/\mathbb{F}_q$ be a scheme, $\mathscr{L}$ a line bundle over S and let $\mathbb{G}_{a/\mathscr{L}}$ be the additive group scheme corresponding to the line bundle $\mathscr{L}$. For all open subsets $U \subset S$ the group scheme is defined by

$$\mathbb{G}_{a/\mathscr{L}}(U) = \mathscr{L}(U).$$

The (additive) groups $\mathscr{L}(U)$ are in a canonical way $\mathbb{F}_q$-vector spaces.

Proposition 3.1

1) Let $\mathscr{L}$ and $\mathscr{M}$ be line bundles over S. Let $\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/\mathscr{M}})$ be the $\mathbb{F}_q$-linear endomorphisms of the corresponding group schemes. Let $\mathscr{H}om_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/\mathscr{M}})$ be the associated homomorphism sheaf over S. Then it follows that

$$\mathscr{H}om_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/\mathscr{M}}) \cong \bigoplus_{n=0}^{\infty} \mathscr{M} \otimes_{\mathcal{O}_S} \mathscr{L}^{-q^n}.$$

2) Let $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})$ be the ring of $\mathbb{F}_q$-linear endomorphism of the group scheme $\mathbb{G}_{a/\mathscr{L}}$ and let $\mathscr{E}nd_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})$ be the associated endomorphism sheaf over S. We have

$$\mathscr{E}nd_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}) \cong \bigoplus_{n=0}^{\infty} \mathscr{L}^{1-q^n}.$$

Proof Cf. [Leh00], chapter 1, proposition 2.3, page 5. □

Corollary 3.2
Let $U = \operatorname{spec} R \subseteq S$ *be an affine, open trivialization of the line bundle* $\mathscr{L}$. *It follows*

$$\mathscr{E}nd_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})(U) \cong \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a/R}) \cong R\{\tau\}.$$

Remark 3.3
A small introduction into the theory of skew polynomial rings over a field can be found in the article [Thi01], section 2.3.

Remark 3.4
In the setting of proposition 3.1 let $\varphi \in \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})$ be a *finite* morphism. Then for all $s \in S$ the rank (rk) of φ is well defined. The rank of φ is constant on all connecting components of S. (Cf. [Leh00], chapter 1, proposition 2.6, page 6).

Definition 3.5 ([Dri76])
Let $\operatorname{char} : S \longrightarrow \operatorname{spec} A$ *be a morphism over* $\mathbb{F}_q$. *A* Drinfeld-module $E := (\mathbb{G}_{a/\mathscr{L}}, e)$ *consits of an additive group scheme* $\mathbb{G}_{a/\mathscr{L}}$ *and a ring homomorphism* $e : A \longrightarrow \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})$ *such that:*

1) *The morphism* $e(a)$ *is finite for all* $a \in A$ *and for all points of* S *there exists an element* $a \in A$ *such that* $\operatorname{rk} e(a) > 1$.

2) *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{\;e\;} & \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}) \\ & \searrow^{\operatorname{char}} \quad \swarrow_{\partial} & \\ & \mathcal{O}_S(S) & \end{array}$$

commutes.

Proposition 3.6
1) *Let* S *be a connected scheme. Then there is a natural number* $d > 0$ *such that*

$$\operatorname{rk} e(a) = -d \deg(\infty)\infty(a).$$

The number d *is called the rank of a Drinfeld module. If the rank of a Drinfeld module is constant on all connected components then we can define the rank of a Drinfeld module to be* d.

2) *Morphisms of Drinfeld modules* $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ *and* $F = (\mathbb{G}_{a/\mathscr{L}'}, f)$ *are homomorphism of group schemes over* $\mathbb{F}_q$

$$u : E \longrightarrow F$$

such that $ue(a) = f(a)u$ *for all* $a \in A$ *in particular the diagram*

$$\begin{array}{ccc} E & \xrightarrow{e(a)} & E \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ F & \xrightarrow[f(a)]{} & F \end{array}$$

commutes.

3) *Let* $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ *be a Drinfeld module of rank* d. *The there exists an isomorphism* u *of Drinfeld modules such that*

$$ue(a)u^{-1} =: f(a),$$

and such that for all affine, open trivializations $\operatorname{spec} R \subseteq S$ *of* E *we have*

$$f(a) = \sum_{i=0}^{-d\deg(\infty)\infty(a)} r_i(a)\tau^i$$

and the leading coefficient $r_{-d\deg(\infty)\infty(a)}(a)$ *is a unit in* R.

A Drinfeld module of the above form is called standard. *If we require in addition that* $\partial u = 1$ *then* u *is unique ([Leh00] chapter 1, 2.8, chapter 2, 2.3).*

4) *If* $S = \operatorname{spec} R$ *is an affine scheme and if* $\mathscr{L}$ *is trivial then we have*

$$\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}) \cong R\{\tau\}$$

and we get the diagram:

$$\begin{array}{ccc} A & \xrightarrow{\quad e \quad} & R\{\tau\} \\ & {\scriptstyle \mathrm{char}}\searrow \quad \swarrow{\scriptstyle \partial} & \\ & R & \end{array}$$

In this case we call the Drinfeld module simply (R, e).

5) *We define* $\deg_\tau$ *to be the degree of an element in the skew polynomial ring* $R\{\tau\}$. *Then if the leading coefficient of an element* $e(\tau) \in R\{\tau\}$ *is invertible the we can perform the right division algorithm*

$$F(\tau) = H(\tau)e(\tau) + R(\tau) \text{ such that } \deg_\tau e(\tau) > \deg_\tau R(\tau).$$

3.2 R $\otimes$ A module structure on R$\{\tau\}$

Let (R, e) be a Drinfeld module over A. On $R\{\tau\}$ we can define a $R \otimes A$ module structure by

$$(\lambda \otimes a)F(\tau) := \lambda F(\tau)e(a).$$

Lemma 3.7
Let (R, e) and (R, f) be Drinfeld modules and let $u : (R, e) \longrightarrow (R, f)$ be an isomorphism. Then the induced $R \otimes A$ module structures on $R\{\tau\}$ are isomorphic.

Proof The isomorphism u is represented by an invertible element $u(\tau) \in R\{\tau\}$. In particular there exists an element $u(\tau)^{-1} \in R\{\tau\}$ such that $u(\tau)u(\tau)^{-1} = 1$. Let

$$\begin{aligned} \varphi : R\{\tau\} &\longrightarrow R\{\tau\} \\ F(\tau) &\longmapsto F(\tau)u(\tau)^{-1} \end{aligned}$$

be the map given by right multiplication with $u(\tau)^{-1}$. We get

$$\begin{aligned} \varphi((r \otimes a)F(\tau)) &= rF(\tau)e(a)u(\tau)^{-1} = rF(\tau)u(\tau)^{-1}f(a) \\ &= (r \otimes a)\varphi(F(\tau)). \end{aligned}$$

□

3.3 Projektivity of R$\{\tau\}$

Proposition 3.8
Let (R, e) be a Drinfeld module of rank d over A. Then $R\{\tau\}$ is as a $R \otimes A$ module finitely generated and locally free of rank d.

Because of lemma 3.7 and proposition 3.3.3 it is sufficient to show the assumption for standard Drinfeld modules.
From the definition of Drinfeld modules there exists an element $a \in A$ such that $\deg_\tau(e(a)) > 0$. The right division algorithm with $e(a)$ in the ring $R\{\tau\}$ gives us as system of generators $1, \ldots, \tau^{\deg_\tau(e(a))-1}$ of $R\{\tau\}$ as $R \otimes A$ module. In particular $R\{\tau\}$ is a finitely generated $R \otimes A$-module.
We will proof proposition 3.8 in five consecutive steps:

3.3.1 $R = L$ is a field and $A = \mathbb{F}_q[T]$.

3.3.2 $R = L$ is a field.

3.3.3 R is a reduced noetherian ring.

3.3.4 R is a noetherian ring.

3.3.5 R is an arbitrary commutative ring.

3.3.1 $\mathbf{R = L}$ is a field and $\mathbf{A = \mathbb{F}_q[T]}$

In this case $L \otimes \mathbb{F}_q[T] \cong L[T]$ is a principal ideal domain. If $f(T) \in L[T]$ and $F(\tau) \in L\{\tau\}$ then we have

$$\deg_\tau(f(T)F(\tau)) = \deg_\tau F(\tau) + d\deg(f(T)).$$

We show that the elements $1, \ldots, \tau^{d-1}$ form a $L[T]$-base of $L\{\tau\}$. Right division algorithm by $e(T)$ shows that the elements form a system of generators. If

$$\sum_{i=0}^{d-1} f_i(T)\tau^i = 0$$

is a zero representation then we have $\deg_\tau(f_i(T)\tau^i) = i + d\deg(f_i(T))$. As $0 \leq i < d$ only the trivial representation is possible.

3.3.2 $\mathbf{R = L}$ is a field

There exists an element $T \in A$ such that F is a finite separable field extension of $\mathbb{F}_q(T)$ and such that A is the integral closure of $\mathbb{F}_q[T]$ in F (cf. [Wie98], section 2.2, page 5 and [Koc00], section 5.1, page 142). From proposition 2.2, 1) and 4), corollary 2.25 and proposition 2.20 we know that $L \otimes A$ is the integral closure of $L[T]$ in the field of fractions $\mathrm{Quot}(L \otimes A)$.
The restriction of the Drinfeld module from A to $\mathbb{F}_q[T]$ gives a Drinfeld module over $\mathbb{F}_q$ of rank $-d\deg(\infty)\infty(T)$ and the $L[T]$ module $L\{\tau\}$ is free of this rank by 3.3.1.
Let $m \in L\{\tau\}$ be a torsion element in particular there exists an element $y \in L \otimes A$ such that $ym = 0$. Because y is integral over $L[T]$ there exists an equation $\sum_{i=0}^{n-1} x_i y^i + y^n = 0$ such that $x_i \in L[T]$. As $L \otimes A$ is a domain we can assume without loss of generality that $x_0 \neq 0$. We conclude

$$\begin{aligned} 0 = \; & -y^n m = \sum_{i=0}^{n-1} x_i y^i m \\ & = x_0 m. \end{aligned}$$

As the operation of $L[T]$ on $L\{\tau\}$ is torsion free we must have $m = 0$.
From proposition 2.5 and proposition 2.10 we conclude that $L\{\tau\}$ is a projective module of finite rank over $L \otimes A$.

As

$$[L \otimes A : L[T]] \operatorname{rang}_{L\otimes A} L\{\tau\} = \operatorname{rang}_{L[T]} L\{\tau\}.$$

it is sufficient to calculate the rank of

$$[L \otimes A : L[T]] = [A : \mathbb{F}_q[T]].$$

Let ∞' be the restriction of the valuation ∞ on $\mathbb{F}_q(T)$. As $x(T) \geq 0$ for all valuations $x \neq \infty$ the extension is unramified in ∞' in particular ∞ is the unique valuation lying over ∞'. From the *fundamental equation of algebraic number theory* ([Neu92] chapter 2, proposition 8.5) we conclude

$$[F : \mathbb{F}_q(T)] = [A : \mathbb{F}_q[T]] = -\infty(T) \deg(\infty).$$

This proofs the assumption.

3.3.3 R is a reduced noetherian ring

We use the criterion for projectivity for reduced rings (cf. proposition 2.7).
Let $\tilde{\mathfrak{p}} \in \operatorname{spec}(R \otimes A)$, let $\tilde{\pi} : \operatorname{spec}(R \otimes A) \longrightarrow \operatorname{spec} R$ be the canonical projection and define $\mathfrak{p} := \tilde{\pi}(\tilde{\mathfrak{p}})$. If $k(\mathfrak{p})$ is the associated residue field of $\mathfrak{p}$ then we have $\tilde{\mathfrak{p}} \in \operatorname{spec}(k(\mathfrak{p}) \otimes A)$. If $\rho : \operatorname{spec}(k(\mathfrak{p}) \otimes A) \longrightarrow \operatorname{spec}(R \otimes A)$ is the fiber of $\mathfrak{p}$ then $\rho^* R\{\tau\} \cong k(\mathfrak{p})\{\tau\}$ as a $k(\mathfrak{p}) \otimes A$ module. It is locally free of rank d by 3.3.2. We conclude $\dim_{k(\tilde{\mathfrak{p}})} k(\tilde{\mathfrak{p}}) \otimes_{R\otimes A} R\{\tau\} = d$. As $\tilde{\mathfrak{p}}$ is arbitrary the assumptions of 2.7 are satisfied.

3.3.4 R is a noetherian ring

We use theorem 2.8 to get into the setting of 3.3.3.
Let R be an arbitrary ring and let $\mathfrak{n}$ be its nilradical. With $\mathfrak{n}$ also $\mathfrak{n} \otimes A$ is a nilpotent ideal in $R \otimes A$. We have $R\{\tau\}/(\mathfrak{n} \otimes A)R\{\tau\} \cong (R/\mathfrak{n})\{\tau\}$ and $R \otimes A/(\mathfrak{n} \otimes A) \cong R/\mathfrak{n} \otimes A$. As $R\{\tau\}$ is a free R module moreover

$$(\mathfrak{n} \otimes A) \otimes_{R\otimes A} R\{\tau\} \cong \mathfrak{n} \otimes_R R\{\tau\} \cong \mathfrak{n}R\{\tau\}.$$

As by 3.3.3 the assumptions of theorem 2.8 for the reduced ring $R/\mathfrak{n}$ are statisfied the module $R\{\tau\}$ is flat. Because it is finitely generated we conclude it is also projektive.

3.3.5 R is an arbitrary commutative ring

As A is a finitely generated $\mathbb{F}_q$-algebra there exists a finitely generated $\mathbb{F}_q$-subalgebra R' of R such that the image of A lies in $R'\{\tau\}$. From 3.3.4 we know that $R'\{\tau\}$ is a locally free $R' \otimes A$ module of rank d. The assumption follows by extension of the ground ring.

3.4 Konstruction of the fiber at ∞

Let (R,e) be a standard Drinfeld module of rank d over the affine scheme $S = \operatorname{spec} R$.
Below we will apply the construction of vector bundles from section 2.8. By proposition 3.8 we know that $R\{\tau\}$ is a projektive $R \otimes A$-module of rank d. We are looking for a projektive $R \otimes \mathcal{O}_{X,\infty}$-module $\mathscr{E}_\infty$ of rank d and an isomorphism

$$\mathscr{E}_\infty \otimes_{\mathcal{O}_{X,\infty}} F \xrightarrow{\sim} R\{\tau\} \otimes_A F.$$

Using this isomorphism we can regard $\mathscr{E}_\infty$ as a subset of $R\{\tau\} \otimes_A F$.
By building the common denominator all elements of $R\{\tau\} \otimes_A F$ are of the form $F(\tau) \otimes x$ such that $F(\tau) \in R\{\tau\}$ and $x \in F$. We define a continuation of the natural degree map on $R\{\tau\}$ by

$$\begin{aligned} \deg_\tau : R\{\tau\} \otimes_A F &\longrightarrow \mathbb{Z} \cup \{-\infty\} \\ F(\tau) \otimes x &\longmapsto \deg_\tau F(\tau) - d\deg(\infty)\infty(x). \end{aligned}$$

This is well defined because for all $a \in A$ we have

$$\deg_\tau F(\tau)e(a) = \deg_\tau F(\tau) + \deg_\tau e(a).$$

(The leading coefficient of $e(a)$ is a unit in R.) There is the strong triangle inequality:

$$\deg_\tau(\alpha + \beta) \leq \max(\deg_\tau \alpha, \deg_\tau \beta)$$

and $F(\tau) \otimes x = 0$ iff $F(\tau) = 0 \vee x = 0$.

Definition 3.9 ([BS97])
For all $i \in \mathbb{Z}$ we define $\mathscr{E}_{i,\infty} := \{\alpha \in R\{\tau\} \otimes_A F \mid \deg_\tau \alpha \leq i\}$.

Let $r \otimes x \in R \otimes \mathcal{O}_{X,\infty}$ (that is $\infty(x) \geq 0$) and $F(\tau) \otimes y \in \mathscr{E}_{i,\infty}$. Then

$$\begin{aligned} \deg_\tau((r \otimes x)F(\tau) \otimes y) &= \deg_\tau(rF(\tau) \otimes xy) \\ &\leq \deg_\tau(F(\tau) \otimes y) - d\deg(\infty)\infty(x) \\ &\leq \deg_\tau(F(\tau) \otimes y) \leq i \end{aligned}$$

In particular $\mathscr{E}_{i,\infty}$ is a $R \otimes \mathcal{O}_{X,\infty}$-module for all $i \in \mathbb{Z}$.

3.5 Projektivitiy of the modules $\mathscr{E}_{i,\infty}$

Proposition 3.10
For all $i \in \mathbb{Z}$ the module $\mathscr{E}_{i,\infty}$ is a finitely generated, projektive $R \otimes \mathcal{O}_{X,\infty}$-module of rank d.

We show first that $\mathscr{E}_{i,\infty}$ is finitely generated. Then we proof proposition 3.10 sukczessive for the following four cases:

3.5.1 $R = L$ is a field.

3.5.2 R is a reduced noetherian ring.

3.5.3 R is a noetherian ring.

3.5.4 R is an arbitrary commutative ring.

Lemma 3.11
Let $a \in A$, let be $k := \deg_\tau e(a) > 0$ and let $\pi_\infty \in F$ such that $\infty(\pi_\infty) = 1$. Then it follows that the $R \otimes \mathcal{O}_{X,\infty}$-module $\mathscr{E}_{i,\infty}$ is generated by the elements

$$\left\{ \tau^r \otimes \pi_\infty^s \;\middle|\; \begin{array}{c} 0 \le r < k \\ \dfrac{-i}{d\deg(\infty)} \le s \le \dfrac{-i}{d\deg(\infty)} + \dfrac{k-1}{d\deg(\infty)} + 1 \\ r - d\deg(\infty)s \le i \end{array} \right\}.$$

In particular the $R \otimes \mathcal{O}_{X,\infty}$-modules $\mathscr{E}_{i,\infty}$ are finitely generated for all $i \in \mathbb{Z}$.

Proof (The numbering relates to the inequalities.)

1) Let $F(\tau) \otimes x \in \mathscr{E}_{i,\infty}$. If $\deg_\tau F(\tau) \ge k$ the division algorithm gives $F(\tau) = H(\tau)e(a) + R(\tau)$ such that $\deg_\tau H(\tau)e(a) = \deg_\tau F(\tau)$ and $\deg_\tau R(\tau) < k$. We conclude

$$\begin{aligned} F(\tau) \otimes x &= H(\tau)e(a) \otimes x + R(\tau) \otimes x \\ &= H(\tau) \otimes ax + R(\tau) \otimes x. \end{aligned}$$

Both summands are elements of $\mathscr{E}_{i,\infty}$. From the assumption we know $\deg_\tau R(\tau) < k$. As $\deg_\tau H(\tau) < \deg_\tau F(\tau)$ we can do induction over $\deg_\tau F(\tau)$. We see that the $R \otimes \mathcal{O}_{X,\infty}$-modul $\mathscr{E}_{i,\infty}$ is generated by elements of the form $\tau^r \otimes x$ with $0 \le r < k$.

2) All $x \in F$ are representable as $x = \varepsilon \pi_\infty^s$ such that $\varepsilon \in \mathcal{O}_{X,\infty}^\star$ and $s = \infty(x)$. We conclude that the element $F(\tau) \otimes x \in \mathscr{E}_{i,\infty}$ is in the set generated by $\tau^r \otimes \pi_\infty^s$ such that $r - d\deg(\infty)s \le i$.

3) From $0 \le r < k$ we conclude that $s \ge -\frac{i}{d\deg(\infty)}$.

4) If $r - d\deg(\infty)(s-1) \le i$ then we have

$$\tau^r \otimes \pi_\infty^s = (1 \otimes \pi_\infty)\tau^r \otimes \pi_\infty^{s-1} \text{ mit } \tau^r \otimes \pi_\infty^{s-1} \in \mathscr{E}_{i,\infty}.$$

That is $\tau^r \otimes \pi_\infty^s$ is in the set generated by $\tau^r \otimes \pi_\infty^{s-1}$. As $r < k$ this is in particular true for $s \ge \frac{-i}{d\deg(\infty)} + \frac{k-1}{d\deg(\infty)} + 1$. □

Lemma 3.12
For all $i \in \mathbb{Z}$ the R-module $\mathscr{E}_{i,\infty}/\mathscr{E}_{i-1,\infty}$ is free of rank 1.
If $\tau^r \otimes \pi_\infty^s \in L\{\tau\} \otimes_A F$ such that $r - d\deg(\infty)s = i$ and $\deg_\tau \tau^r \otimes \pi_\infty^s = i$ then $\tau^r \otimes \pi_\infty^s$ is a base element of $\mathscr{E}_{i,\infty}/\mathscr{E}_{i-1,\infty}$ as a R-module.

Proof We show first that $\tau^r \otimes \pi_\infty^s$ is a generator. Let be $F(\tau) \otimes x \in \mathscr{E}_{i,\infty} \setminus \mathscr{E}_{i-1,\infty}$ such that $F(\tau) = \sum_{t=0}^n \rho_n \tau^n$ and $\rho_n \neq 0$. We conclude $F(\tau) \otimes x \equiv \rho_n \tau^n \otimes x \mod \mathscr{E}_{i-1,\infty}$. Let $\lambda \in F$ be such that $\lambda \pi_\infty^s = x$. Then there exists $a, b \in A$ such that $\lambda = \frac{a}{b}$. Let

$$y := \frac{\pi_\infty^s}{b} = \frac{x}{a}.$$

It follows

$$\begin{aligned} \tau^r \otimes \pi_\infty^s &= \tau^r \otimes \frac{b}{b}\pi_\infty^s = \tau^r e(b) \otimes y \\ &\equiv \rho' \tau^{r+\deg_\tau(e(b))} \otimes y \mod \mathscr{E}_{i-1,\infty} \\ \rho_n \tau^n \otimes x &= \rho_n \tau^n \otimes \frac{a}{a} x = \rho_n \tau^n e(a) \otimes y \\ &\equiv \rho'' \tau^{n+\deg_\tau(e(a))} \otimes y \mod \mathscr{E}_{i-1,\infty} \end{aligned}$$

such that $\rho', \rho'' \in R$. From the assumption we see that ρ' is a unit. Further on we get

$$\begin{aligned} r + \deg_\tau(e(b)) - d\deg(\infty)\infty(y) &= i \\ n + \deg_\tau(e(a)) - d\deg(\infty)\infty(y) &= i \end{aligned}$$

and

$$m := r + \deg_\tau(e(b)) = n + \deg_\tau(e(a)).$$

As ρ' is a unit there exists $\rho \in R$ such that $\rho\rho' = \rho''$. It follows

$$\rho\rho'\tau^m \otimes y = \rho''\tau^m \otimes y$$

and we conclude

$$\rho(\tau^r \otimes \pi_\infty^s) \equiv F(\tau) \otimes x \mod \mathscr{E}_{i-1,\infty}.$$

If $\rho(\tau^r \otimes \pi_\infty^s) = 0$ then $\rho = 0$ because the operation of R is torsion free. This proofs the lemma. □

Lemma 3.13
For all $\mathfrak{p} \not\supseteq R \otimes \mathfrak{m}_{X,\infty}$ and all $i \in \mathbb{Z}$ we have

$$(\mathscr{E}_{i,\infty})_\mathfrak{p} = (\mathscr{E}_{i+1,\infty})_\mathfrak{p} = (R\{\tau\} \otimes_A F)_\mathfrak{p}.$$

Proof Let $\mathfrak{p} \in \operatorname{spec}(R \otimes \mathcal{O}_{X,\infty})$ be such that $\mathfrak{p} \not\supseteq R \otimes \mathfrak{m}_{X,\infty}$. Then we have $1 \otimes \pi_\infty \notin \mathfrak{p}$ and $1 \otimes \pi_\infty$ is a unit of $(R \otimes \mathcal{O}_{X,\infty})_\mathfrak{p}$. As $\mathscr{E}_{i,\infty} \subseteq \mathscr{E}_{i+1,\infty}$ we conclude $(\mathscr{E}_{i,\infty})_\mathfrak{p} \subseteq (\mathscr{E}_{i+1,\infty})_\mathfrak{p}$. Let $\beta \in \mathscr{E}_{i+1,\infty}$. Then we have $(1 \otimes \pi_\infty)\beta \in \mathscr{E}_{i,\infty}$ and

$$\beta = (1 \otimes \pi_\infty)^{-1}(1 \otimes \pi_\infty)\beta \in (\mathscr{E}_{i,\infty})_\mathfrak{p}.$$

We conclude $(\mathscr{E}_{i+1,\infty})_\mathfrak{p} \subseteq (\mathscr{E}_{i,\infty})_\mathfrak{p}$. By $\bigcup_{i\in\mathbb{Z}} \mathscr{E}_{i,\infty} = R\{\tau\} \otimes_A F$ we see further

$$\bigcup_{i\in\mathbb{Z}} (\mathscr{E}_{i,\infty})_\mathfrak{p} = (\mathscr{E}_{i,\infty})_\mathfrak{p} = (R\{\tau\} \otimes_A F)_\mathfrak{p}.$$

3.5.1 R = L is a field

Let $P \in X$, $P \neq \infty$ and let $B := \Gamma(X \setminus \{P\}, \mathcal{O}_X)$. The rings $L \otimes \mathcal{O}_{X,\infty}$ and $L \otimes F$ are localizations of the Dedekind ring $L \otimes B$ and by proposition 2.20 are itself Dedekind rings (cf. corollary 2.25 and remark 2.4).
From proposition 3.8 we get that $L\{\tau\}$ is a projektive $L \otimes A$-modul. This implies that $L\{\tau\} \otimes_A F$ is projektive and in particular a torsion free $L \otimes F$-modul. Thus the operation of the subring $L \otimes \mathcal{O}_{X,\infty}$ on $L\{\tau\} \otimes_A F$ is torsion free. We get that this operation is torsion free on $\mathscr{E}_{i,\infty}$ for all $i \in \mathbb{Z}$ too.
As the modules $\mathscr{E}_{i,\infty}$ are finitely generated and torsion free over the Dedekind ring $L \otimes \mathcal{O}_{X,\infty}$ they are projective (cf. 2.5).
As $(L \otimes F)_{(0)} = (L \otimes \mathcal{O}_{X,\infty})_{(0)}$ and $(0) \not\supseteq R \otimes \mathfrak{m}_{X,\infty}$ we get by Lemma 3.13 $(\mathscr{E}_{i,\infty})_{(0)} = (L\{\tau\} \otimes_A F)_{(0)}$. The $(L \otimes F)_{(0)}$-module $(L\{\tau\} \otimes_A F)_{(0)}$ is a localization of the $L \otimes F$-module $L\{\tau\} \otimes_A F$. For this it has rank d.

Lemma 3.14
1) *Let R be an arbitrary ring over $\mathbb{F}_q$, let $\mathfrak{a}$ be an ideal and define $\bar{R} := R/\mathfrak{a}$. If (R, e) is a standard Drinfeld module of rank d then $(\bar{R}, \bar{e})$ is also a standard Drinfeld module of rank d.*

   If
$$\bar{\mathscr{E}}_{i,\infty} := \{\bar{\alpha} \in \bar{R}\{\tau\} \otimes_A F \mid \deg_\tau \bar{\alpha} \leq i\},$$
   then the map
$$\begin{aligned} \mathscr{E}_{i,\infty}/(\mathfrak{a} \otimes \mathcal{O}_{X,\infty})\mathscr{E}_{i,\infty} &\longrightarrow \bar{\mathscr{E}}_{i,\infty} \\ F(\tau) \otimes x \mod (\mathfrak{a} \otimes \mathcal{O}_{X,\infty})\mathscr{E}_{i,\infty} &\longmapsto \bar{F}(\tau) \otimes x \end{aligned}$$
   is an isomorphism of $\bar{R} \otimes \mathcal{O}_{X,\infty}$-modules.

2) *Let S be a multiplicative closed subset of R. Define $S \otimes 1 := \{s \otimes 1 \in R \otimes \mathcal{O}_{X,\infty} \mid s \in S\}$. Then $S \otimes 1$ is a multiplicative closed subset and we have*
$$(S \otimes 1)^{-1}(R \otimes \mathcal{O}_{X,\infty}) = S^{-1}R \otimes \mathcal{O}_{X,\infty}.$$

If (R, e) is a standard Drinfeld module of rank d then the canonical extension on $(S^{-1}R, e)$ is a standard Drinfeld module of rank d too. Define

$$\tilde{\mathscr{E}}_{i,\infty} := \{\alpha \in S^{-1}R\{\tau\} \otimes F \mid \deg_\tau \alpha \leq i\},$$

then the map

$$(S \otimes 1)^{-1}\mathscr{E}_{i,\infty} \longrightarrow \tilde{\mathscr{E}}_{i,\infty}$$
$$(s \otimes 1)^{-1}(F(\tau) \otimes x) \longmapsto s^{-1}F(\tau) \otimes x$$

is an isomorphism of $S^{-1}R \otimes \mathcal{O}_{X,\infty}$ modules.

Proof

1) The map is well defined because $\deg_\tau \bar{F}(\tau) \leq \deg_\tau F(\tau)$. If $\tau^r \otimes \pi_\infty^s$ is such that $r - d\deg(\infty)s \leq i$ then $\bar{\tau}^r \otimes \pi_\infty^s$ for $r - d\deg(\infty)s \leq i$ is a system of generators of $\bar{\mathscr{E}}_{i,\infty}$. This implies the surjectivity of the map.

   Let $\bar{F}(\tau) \otimes x = 0$. By $x \neq 0$ we conclude $\bar{F}(\tau) = 0$. In particular $F(\tau) \in \mathfrak{a}\{\tau\}$. This implies the injectivity of the map.

2) The proof of part 2) is similar to part 1). □

3.5.2 R is a reduced noetherian ring

If $\tilde{y} \in \operatorname{spec}(R \otimes \mathcal{O}_{X,\infty})$ then there exists $y \in \operatorname{spec} R$ such that $\tilde{y}$ is in the fiber over y with respect to the projection $\pi : \operatorname{spec}(R \otimes \mathcal{O}_{X,\infty}) \longrightarrow \operatorname{spec} R$. If $k(y)$ is the residue field of y then $\tilde{y} \in \operatorname{spec}(k(y) \otimes \mathcal{O}_{X,\infty})$. If $\rho : \operatorname{spec}(k(y) \otimes \mathcal{O}_{X,\infty}) \longrightarrow \operatorname{spec}(R \otimes \mathcal{O}_{X,\infty})$ is the fiber over y then $\rho^*\mathscr{E}_{i,\infty}$ is the $k(y) \otimes \mathcal{O}_{X,\infty}$-module $\tilde{\bar{\mathscr{E}}}_{i,\infty}$ for an appropiate ideal $\mathfrak{p}$ in R and an appropiate multiplicative closed subset S (cf. lemma 3.14, 1 and 2). It is by 3.5.1 locally free of rank d. This implies $\dim_{k(\tilde{y})} k(\tilde{y}) \otimes_{R \otimes \mathcal{O}_{X,\infty}} \mathscr{E}_{i,\infty} = d$. As $\tilde{y}$ is arbitrary the assumptions of proposition 2.7 are satisfied.

3.5.3 R is a noetherian ring

As in the proof of 3.8, section 3.3.4 we must show that the canonical map $\mathfrak{n} \otimes_R \mathscr{E}_{i,\infty} \longrightarrow \mathfrak{n}\mathscr{E}_{i,\infty}$ for the nilradical $\mathfrak{n}$ of R is injektiv. We show even more that $\mathscr{E}_{i,\infty}$ is a flat R-module for all $i \in \mathbb{Z}$.
As $R\{\tau\}$ is a projective $R \otimes A$ modul we get that $R\{\tau\} \otimes F$ is a projective $R \otimes F$ modul too. On the other hand $R \otimes F$ is a projective (even free) R modul. We get that $R\{\tau\} \otimes F$ is a projektive and even a flat R modul.
Let

$$0 \longrightarrow \mathscr{E}_{i,\infty} \longrightarrow \mathscr{E}_{i+1,\infty} \longrightarrow \mathscr{E}_{i+1,\infty}/\mathscr{E}_{i,\infty} \longrightarrow 0$$

be the canonical exact sequence and let $\mathfrak{a}$ be an ideal in R. Tensorizing the sequence by $\mathfrak{a}$ gives

$$\begin{aligned} &\longrightarrow \mathrm{Tor}^1(\mathfrak{a}, \mathscr{E}_{i+1,\infty}/\mathscr{E}_{i,\infty}) \longrightarrow \mathfrak{a} \otimes_R \mathscr{E}_{i,\infty} \longrightarrow \\ &\mathfrak{a} \otimes_R \mathscr{E}_{i+1,\infty} \longrightarrow \mathfrak{a} \otimes_R \mathscr{E}_{i+1,\infty}/\mathscr{E}_{i,\infty} \longrightarrow 0. \end{aligned}$$

By 3.12 we see that $\mathscr{E}_{i+1,\infty}/\mathscr{E}_{i,\infty}$ is a projective R-module and the torsion group is zero. This shows that the map

$$\mathfrak{a} \otimes_R \mathscr{E}_{i,\infty} \longrightarrow \mathfrak{a} \otimes_R \mathscr{E}_{i+1,\infty}$$

is injective.
If $K_i := \mathrm{Ker}(\mathfrak{a} \otimes_R \mathscr{E}_{i,\infty} \longrightarrow \mathfrak{a}\mathscr{E}_{i,\infty})$ then we conclude form above that the natural map $K_i \longrightarrow K_{i+1}$ is injectiv too.
If

$$0 \longrightarrow K_i \longrightarrow \mathfrak{a} \otimes_R \mathscr{E}_{i,\infty} \longrightarrow \mathfrak{a}\mathscr{E}_{i,\infty} \longrightarrow 0$$

is the canonical exact sequence then applying the exact functor $\varinjlim$ gives the exact sequence

$$0 \longrightarrow \varinjlim K_i \longrightarrow \varinjlim(\mathfrak{a} \otimes_R \mathscr{E}_{i,\infty}) \longrightarrow \varinjlim \mathfrak{a}\mathscr{E}_{i,\infty} \longrightarrow 0.$$

Furthermore we have $\varinjlim(\mathfrak{a} \otimes_R \mathscr{E}_{i,\infty}) = \mathfrak{a} \otimes_R \varinjlim \mathscr{E}_{i,\infty} = \mathfrak{a} \otimes_R (R\{\tau\} \otimes F)$. As $R\{\tau\} \otimes F$ is a projective R-module we conclude that $\varinjlim K_i = 0$. The maps in the limit are all injective. For this $K_i = 0$ for all $i \in \mathbb{Z}$. By [Mat86] §7, theorem 7.7, the R-modules $\mathscr{E}_{i,\infty}$ are flat.

3.5.4 R is an arbitrary ring

By 3.11 the module $\mathscr{E}_{i,\infty}$ is a finitely generated $R \otimes \mathcal{O}_{X,\infty}$-module. Let $m_1, \ldots, m_r \in \mathscr{E}_{i,\infty} \subseteq R\{\tau\} \otimes_A F$ be generators. Then there exists a finitely generated $\mathbb{F}_q$-subalgebra R' of R such that $m_1, \ldots, m_r$ are elements of $R'\{\tau\} \otimes_A F$. Let $\mathscr{E}'_{i,\infty}$ be to the ring R' associated submodule of $R'\{\tau\} \otimes_A F$. The canonical map

$$R \otimes_{R'} \mathscr{E}'_{i,\infty} \longrightarrow \mathscr{E}_{i,\infty}$$

is in this case an isomorphism. As $\mathscr{E}_{i,\infty}$ is an extension of scalars of the module $\mathscr{E}'_{i,\infty}$ we conclude the assumption. This proofs proposition 3.10 completeley.

3.6 Construction of the vector bundles

By the results of section 3.3 and 3.5 we will construct $\mathcal{O}_{X \times S}$-vector bundles for all $i \in \mathbb{Z}$. Furthermore we will get canonical maps

$$s_i : \mathscr{E}_i \longrightarrow \mathscr{E}_{i+1}$$

and

$$t_i : {}^{\tau}\mathscr{E}_i \longrightarrow \mathscr{E}_{i+1}$$

for all $i \in \mathbb{Z}$.

Lemma 3.15
Let be $\pi_\infty \in F$ such that $\infty(\pi_\infty) = 1$. Then the map

$$\begin{aligned} \mathscr{E}_{i,\infty} &\longrightarrow \mathscr{E}_{i+d\deg(\infty),\infty} \\ \alpha &\longmapsto \alpha\pi_\infty^{-1} \end{aligned}$$

is an isomorphism of $R \otimes \mathcal{O}_{X,\infty}$-modules. The inverse map is given by

$$\begin{aligned} \mathscr{E}_{i+d\deg(\infty),\infty} &\longrightarrow \mathscr{E}_{i,\infty} \\ \alpha &\longmapsto \alpha\pi_\infty. \end{aligned}$$

Proof We get the assumption from the validity of the equation

$$\deg_\tau(\alpha x) = \deg_\tau \alpha - d\deg(\infty)\infty(x)$$

for all $\alpha \in R\{\tau\} \otimes_A F$ and $x \in F$. □

Let $F(\tau) \otimes x \in R\{\tau\} \otimes_A F$ be of degree i. Then we have $\deg_\tau(\tau F(\tau) \otimes x) \leq i+1$. This induces a map

$$\begin{aligned} \mathscr{E}_{i,\infty} &\longrightarrow \mathscr{E}_{i+1,\infty} \\ F(\tau) \otimes x &\longmapsto \tau F(\tau) \otimes x \end{aligned}$$

for all $i \in \mathbb{Z}$. It is $\mathcal{O}_{X,\infty}$-linear and q-linear in R. By proposition 2.75 the map factorizes over ${}^{\tau}\mathscr{E}_{i,\infty}$. We conclude:

Lemma 3.16
Let $\mathfrak{p}$ be a prime ideal of $R \otimes \mathcal{O}_{X,\infty}$ such that $\mathfrak{p} \supseteq R \otimes \mathfrak{m}_{X,\infty}$. Then

$$\begin{aligned} ({}^{\tau}\mathscr{E}_{i,\infty})_{\mathfrak{p}} &\longrightarrow (\mathscr{E}_{i+1,\infty})_{\mathfrak{p}} \\ 1 \otimes \alpha &\longmapsto \tau\alpha \end{aligned}$$

is an isomorphism of $(R \otimes \mathcal{O}_{X,\infty})_{\mathfrak{p}}$-modules.

Proof The module $(\mathscr{E}_{i,\infty})_{\mathfrak{p}}$ is a finitely generated $(R\otimes\mathcal{O}_{X,\infty})_{\mathfrak{p}}$-module and we have $R\otimes\mathfrak{m}_{X,\infty} \subseteq \mathrm{Rad}((R\otimes\mathcal{O}_{X,\infty})_{\mathfrak{p}}) = \mathfrak{p}$. By lemma 3.11 $\mathscr{E}_{i,\infty}$ is generated by $\tau^r \otimes \pi_\infty^s$ with r, s as in 3.11. Let $M \subseteq (\mathscr{E}_{i+1,\infty})_{\mathfrak{p}}$ be the submodule generated by $\tau^{r+1} \otimes \pi_\infty^s$. By the lemma of Nakayama (cf. [Mat86], corollary above theorem 2.2) we must show

$$(\mathscr{E}_{i+1,\infty})_\mathfrak{p} = M + (R \otimes \mathfrak{m}_{X,\infty})(\mathscr{E}_{i+1,\infty})_\mathfrak{p}.$$

As $\tau^r \otimes \pi_\infty^s$ is a system of generators of $(\mathscr{E}_{i+1,\infty})_\mathfrak{p}$ and the elements $\tau^r \otimes \pi_\infty^s$ are elements of M for $r > 0$ it is sufficient to show that $1 \otimes \pi_\infty^s \in (R \otimes \mathfrak{m}_{X,\infty})(\mathscr{E}_{i+1,\infty})_\mathfrak{p}$. We regard the following two cases:

Case I $-d \deg \infty (s-1) \leq i+1$. Then

$$1 \otimes \pi_\infty^s = (1 \otimes \pi_\infty) 1 \otimes \pi_\infty^{s-1} \in (R \otimes \mathfrak{m}_{X,\infty})(\mathscr{E}_{i+1,\infty})_\mathfrak{p}$$

Case II $-d \deg \infty (s-1) > i+1$. Then

$$\begin{aligned} 1 \otimes \pi_\infty^s &= 1 \otimes \pi_\infty^s \frac{a}{a} \\ &= e(a) \otimes \frac{\pi_\infty^s}{a} \\ &= \sum_{n=0}^{k} r_n \tau^n \otimes \frac{\pi_\infty^s}{a}. \end{aligned}$$

If $n > 0$ then the summands are elements of M. If $r_0 = 0$ then nothing is to show. Otherwise we have $-(s - \frac{k}{d \deg \infty}) d \deg(\infty) \leq i+1$ in particular $-(s-1) d \deg(\infty) \leq i+1$ and we can use case I. □

Lemma 3.17
Let $\pi_\infty \in F$ be such that $\infty(\pi_\infty) = 1$. Then the map

$$\begin{aligned} \mathscr{E}_{i,\infty} \otimes_{\mathcal{O}_{X,\infty}} \pi_\infty^{-1} \mathcal{O}_{X,\infty} &\longrightarrow \mathscr{E}_{i+d \deg \infty, \infty} \\ \alpha \otimes \pi_\infty^{-1} x &\longmapsto \alpha \pi_\infty^{-1} x \end{aligned}$$

is an isomorphism of $R \otimes \mathcal{O}_{X,\infty}$-modules. The twist by $\pi_\infty \mathcal{O}_{X,\infty}$ gives an isomorphis with $\mathscr{E}_{i-d \deg \infty, \infty}$.

Proof Let $\alpha \in \mathscr{E}_{i+d \deg \infty, \infty}$. Then $\alpha \pi_\infty \in \mathscr{E}_{i,\infty}$ is in the preimage. As with $\mathscr{E}_{i,\infty}$ also $\mathscr{E}_{i,\infty} \otimes_{\mathcal{O}_{X,\infty}} \pi_\infty^{-1} \mathcal{O}_{X,\infty}$ is projective of rank d. As surjective maps of projective modules of equal rank are isomorphisms we conclude the first assumption. The proof of the second assumption is similar. □

Remark 3.18
Let (R, e) be an arbitrary Drinfeld module over A. By proposition 3.6, 3) there exists a standard Drinfeld module (R, f) and a unique isomorphism $u : (R, e) \longrightarrow (R, f)$ of Drinfeld modules such that $\partial u = 1$.

By lemma 3.7 u induces an isomorphism between the two $R \otimes A$-module structures on $R\{\tau\}$. In particular we gain an isomorphism

$$\begin{array}{ccc} R\{\tau\} \otimes_{A,e} F & \xrightarrow{\sim} & R\{\tau\} \otimes_{A,f} F \\ F(\tau) \otimes x & \longmapsto & F(\tau)u^{-1} \otimes x \end{array}$$

of $R \otimes F$-modules.
If (R, e) is an arbitrary Drinfeld module we define $R \otimes \mathcal{O}_{X,\infty}$-modules $\mathscr{E}_{i,\infty} \subseteq R\{\tau\} \otimes_{A,e} F$ to be the preimages of the modules $\mathscr{E}_{i,\infty} \subseteq R\{\tau\} \otimes_{A,f} F$ defined in the standard case. In particular we can use all properties from the standard case.

Remark 3.19
By proposition 2.69, proposition 3.8 and proposition 3.10 we will construct for all $i \in \mathbb{Z}$ the $\mathcal{O}_{X \times S}$ vector bundles $\{\mathscr{E}_i\}_{i \in \mathbb{Z}}$ of rank d.
For all $i \in \mathbb{Z}$ we get from the inclusion of the modules $\mathscr{E}_{i,\infty} \hookrightarrow \mathscr{E}_{i+1,\infty}$ a map

$$s_i : \mathscr{E}_i \longrightarrow \mathscr{E}_{i+1}.$$

Left multiplication by τ defines for all $i \in \mathbb{Z}$ an $\mathcal{O}_S$-q-linear map $\mathscr{E}_{i,\infty} \hookrightarrow \mathscr{E}_{i+1,\infty}$. In particular by proposition 2.75 we get for all $i \in \mathbb{Z}$ a map

$$t_i : {}^{\tau}\mathscr{E}_i \longrightarrow \mathscr{E}_{i+1}.$$

Corollary 3.20
Let (R, e) be a standard Drinfeld module. Then we have for all $i \in \mathbb{Z}$

$$\mathscr{E}_i(X \times S) = \{F(\tau) \in R\{\tau\} \mid \deg_\tau F(\tau) \leq i\}.$$

Proof By construction the patching of $\mathscr{E}_{i,\infty}$ and $R\{\tau\}$ are induced by the canonical maps

$$\mathscr{E}_{i,\infty} \longrightarrow R\{\tau\} \otimes_A F \longleftarrow R\{\tau\}.$$

From the definition of $\mathscr{E}_{i,\infty}$ we can now conclude the assumption. □

Corollary 3.21
In the setting of lemma 3.16 we have:

1) $H^0(X \times S, \mathscr{E}_i) = H^0(X \times s, (\mathscr{E}_i)_s) = 0$ *for* $i < 0$ *and* $s \in S$.

2) $H^1(X \times S, \mathscr{E}_i) = H^1(X \times s, (\mathscr{E}_i)_s) = 0$ *for* $i \geq -1$ *and* $s \in S$.

3) $\varinjlim_{i \in \mathbb{Z}} H^0(X \times S, \mathscr{E}_i) = R\{\tau\}$.

Proof

1) Follows by corollary 3.20.

2) By lemma 3.13 the sheave $\mathscr{E}_i/\mathscr{E}_{i-1}$ is supported only in $\operatorname{spec}\mathcal{O}_{X,\infty}\otimes R$ and by construction we get

$$(\mathscr{E}_i/\mathscr{E}_{i-1})\,|_{\operatorname{spec}\mathcal{O}_{X,\infty}\otimes R} \cong \mathscr{E}_{i,\infty}/\mathscr{E}_{i-1,\infty}.$$

By lemma 3.12 for all $i \geq 0$ the element $\tau^i \in R\{\tau\}$ is a global section that generates $\mathscr{E}_{i,\infty}/\mathscr{E}_{i-1,\infty}$. In particular

$$H^0(X\times S,\mathscr{E}_i) \longrightarrow H^0(X\times S,\mathscr{E}_i/\mathscr{E}_{i-1})$$

is a surjectiv map. Furthermore $H^1(X\times S,\mathscr{E}_{i-1}) = 0$ for all $i \geq 0$.

3) Follows by corollary 3.20. □

3.7 Different approach

In place of the explicit construction of the stalks at ∞ we can construct the vector bundles $\mathscr{E}_i$ by a Proj construction.
Let be $\mathcal{S}_X := \bigoplus_{i=0}^{\infty} H^0(X,\mathcal{O}_X(i\infty))$. By proposition 2.36 we have $\operatorname{Proj}\mathcal{S}_X \cong X$. For all $k\in\mathbb{Z}$ we define

$$M_k := \{f(\tau)\in R\{\tau\} \mid \deg_\tau f(\tau) \leq k\}$$

and for all $i\in\mathbb{Z}$ we define

$$\mathcal{M}_i := \bigoplus_{k=0}^{\infty} M_{kd\deg(\infty)+i}.$$

For all $i\in\mathbb{Z}$ the module $\mathcal{M}_i$ has a canonical graduated $R\otimes\mathcal{S}_X$-module structure. We define

$$\mathscr{E}_i' := \mathcal{M}_i^{\sim}$$

for all $i\in\mathbb{Z}$ the associated $\mathcal{O}_{X\times S}$-module given by the Proj construction.

Proposition 3.22
For all $i\in\mathbb{Z}$ the $\mathcal{O}_{X\times S}$-modules $\mathscr{E}_i'$ and $\mathscr{E}_i$ are canonical isomorphic.

Proof Let $U := X\setminus\infty$. We have

$$\mathscr{E}_i'|_{U\times S} = \mathcal{M}_i[e^{-1}]_{(0)} = R\{\tau\} = \mathscr{E}_i|_{U\times S}$$

für all $i \in \mathbb{Z}$. Let $\mathfrak{p}_\infty \subseteq \mathcal{S}_X$ be the homogenous prime ideal associated to the point ∞. We conclude

$$\begin{aligned} \mathscr{E}_i'|_{\operatorname{spec}\mathcal{O}_{X,\infty}\times S} &\cong \mathcal{M}_{i(\mathfrak{p}_\infty)} \\ &\cong \left\{ \frac{f(\tau)}{a} \,\middle|\, \deg_\tau f(\tau) \leq \deg_\tau(e_a) + i,\ 0 \neq a \in A,\ f(\tau) \in R\{\tau\} \right\} \\ &\cong \mathscr{E}_i|_{\operatorname{spec}\mathcal{O}_{X,\infty}\times S} \end{aligned}$$

for all $i \in \mathbb{Z}$. From this we conclude the assumption. □

3.8 Construction over an arbitrary base scheme

Below we will enlarge the construction of the vector bundles for Drinfeld modules over an arbitrary base scheme S. Let (E, e) be a standard Drinfeld module of rank d over S. We define

$$\mathscr{E}_U := \mathscr{H}om_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/S}).$$

As $\mathcal{O}_S \subseteq \mathscr{E}nd_{\mathbb{F}_q}(\mathbb{G}_{a/S})$ there exists a caninical right operation of the structure sheaf $\mathcal{O}_S$ on $\mathscr{E}_U$. By the definition of a Drinfeld module we have the map

$$e : A \longrightarrow \mathscr{E}nd_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})$$

and this induces a canonical left A-module structure on $\mathscr{E}_U$. Both operations are compatible and induce an $\mathcal{O}_{\operatorname{spec} A\times S}$-module structure on $\mathscr{E}_U$.
By proposition 3.1 we have

$$\mathscr{H}om_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/S}) \cong \bigoplus_{n\geq 0} \mathscr{L}^{-q^n}.$$

We define

$$\mathcal{M} := \bigoplus_{k=0}^{\infty} \left(\bigoplus_{n=0}^{kd\deg\infty} \mathscr{L}^{-q^n} \right).$$

Let

$$f : \mathscr{L} \longrightarrow \mathscr{L}^{-q^n} \in \mathscr{E}nd_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}})$$

and let

$$g : \mathcal{O}_S \longrightarrow \mathscr{L}^{-q^m} \in \mathscr{H}om_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/S}),$$

then the composition of the two maps

$$g \circ f : \mathcal{O}_S \longrightarrow \mathscr{L}^{-q^{m+n}} \in \mathscr{H}om_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/S})$$

is well defined and induces a canonical structure of a graduated $\mathcal{S}_X \otimes \mathcal{O}_S$-module on $\mathcal{M}$. By using the Proj functor we get the quasi coherent $\mathcal{O}_{X\times S}$-modul $\mathscr{E}$. The construction of the sheaf $\mathscr{E}$ coincides locally in S with the construction of the sheaf $\mathscr{E}'_0$ in section 3.7. In particular the sheaf $\mathscr{E}$ an $\mathcal{O}_{X\times S}$ is a vector bundle of rank d.
If we use the Proj functor on the $\mathcal{S}_X \otimes \mathcal{O}_S$-modules

$$\mathcal{M}_i := \bigoplus_{k=0}^{\infty} \left(\bigoplus_{n=0}^{i+kd \deg \infty} \mathscr{L}^{-q^n} \right),$$

we get the $\mathcal{O}_{X\times S}$-modules $\mathscr{E}_i$ for all $i \in \mathbb{Z}$. As before they coincide with the construction in section 3.7.

4 Vector bundles of general type

4.1 Definition of vector bundles of general type

Definition 4.1 (Drinfeld, [Dri86])
We call an $\mathcal{O}_{X\times S}$-vector bundle $\mathscr{E}$ of general type *if for all $i \in \mathbb{Z}$*

$$\begin{aligned} \textit{either} \quad & H^0(X \times s, \mathscr{E}(i\infty)_s) = 0 \quad \forall s \in S \\ \textit{or} \quad & H^1(X \times s, \mathscr{E}(i\infty)_s) = 0 \quad \forall s \in S. \end{aligned}$$

Lemma 4.2
1) Let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$-vector bundle of general type. Then for all $k \in \mathbb{Z}$ the sheaves $\mathscr{E}(k\infty)$ are of general type.

2) Let $\mathscr{E}$ be an $\mathcal{O}_{X\times S}$-vector bundle such that

$$\begin{gathered} H^0(X \times s, \mathscr{E}_s) = 0 \quad \forall s \in S \\ \textit{and} \\ H^1(X \times s, \mathscr{E}_s) = 0 \quad \forall s \in S. \end{gathered}$$

then the sheaf $\mathscr{E}$ is of general type.

Proof 1) follows immediately from the definition. Assumption 2) follows from the canonical injection

$$H^0(X \times s, \mathscr{E}(i\infty)_s) \hookrightarrow H^0(X \times s, \mathscr{E}((i+1)\infty)_s)$$

an the canonical surjection

$$H^1(X \times s, \mathscr{E}(i\infty)_s) \twoheadrightarrow H^1(X \times s, \mathscr{E}((i+1)\infty)_s)$$

for all $i \in \mathbb{Z}$ and all $s \in S$. □

4.2 Drinfeld's theorem

Theorem 4.3 (Drinfeld, [Dri86])
The both categories below are equivalent:

I) Category of $\mathcal{O}_{X\times S}$-vector bundles $\mathscr{E}$ of rank $\frac{d}{\deg(\infty)}$, of general type and Euler characteristic $\chi(\mathscr{E}_s) = 0$ for all $s \in S$.

II) Category of $\mathcal{O}_S$-modules $\mathscr{M}$ equipped with an ascending filtration of $\mathcal{O}_S$-modules $\mathscr{M}_i$ and a ring homomorphism $A \longrightarrow \mathrm{End}_{\mathcal{O}_S}(\mathscr{M})$ such that:

a) It is $\varinjlim \mathscr{M}_i = \mathscr{M}$.

b) For all $i \in \mathbb{Z}$ *the* $\mathcal{O}_S$*-modules* $\mathscr{M}_i$ *are vector bundles of rank* $\max(di, 0)$. *Further the* $\mathcal{O}_S$ *modules* $\mathscr{M}_i/\mathscr{M}_{i-1}$ *are vector bundles of rank* d *for* $i > 0$ *and of rank* 0 *for* $i \leq 0$.

c) For all $a \in A$, $a \neq 0$ *it is* $a\mathscr{M}_i \subseteq \mathscr{M}_{i+\deg(a)}$ *and the map*

$$\mathscr{M}_{i+1}/\mathscr{M}_i \xrightarrow{a\cdot} \mathscr{M}_{i+1+\deg(a)}/\mathscr{M}_{i+\deg(a)}$$

is an isomorphism for all $i \geq 0$.

4.3 Category I $\Longrightarrow$ Kategory II

Let $\mathscr{E}$ be an sheaf of $\mathcal{O}_{X\times S}$-modules of general type and for all $i \in \mathbb{Z}$ define $\mathscr{E}_i := \mathscr{E}(i\infty)$. Further we define for all $i \in \mathbb{Z}$ the $\mathcal{O}_S$ modules

$$\mathscr{M}_i := \operatorname{pr}_{S*} \mathscr{E}_i \text{ and } \mathscr{M} := \operatorname{pr}_{S*}(\mathscr{E}|_{\operatorname{spec} A\times S}).$$

Remark 4.4
For $i \leq j$ the canonical inclusions

$$\mathcal{O}_X(i\infty) \hookrightarrow \mathcal{O}_X(j\infty)$$

define an inductive system of maps

$$\mathscr{E}_i \hookrightarrow \mathscr{E}_j.$$

For this using the funktor pr_{S*} induces injective maps

$$\mathscr{M}_i = \operatorname{pr}_{S*} \mathscr{E}_j \hookrightarrow \operatorname{pr}_{S*} \mathscr{E}_j = \mathscr{M}_j.$$

They define an inductive system.

Lemma 4.5
For all $i \in \mathbb{Z}$ *it is* $\mathscr{E}_i|_{\operatorname{spec} A\times S} = \mathscr{E}|_{\operatorname{spec} A\times S}$.

Proof Let $\iota : \operatorname{spec} A \times S \longrightarrow X \times S$ be the canonical embedding. It is

$$\begin{aligned}
\mathscr{E}_i|_{\operatorname{spec} A\times S} &\cong \mathscr{E}|_{\operatorname{spec} A\times S} \otimes_{\mathcal{O}_{\operatorname{spec} A\times S}} (\operatorname{pr}_X^* \mathcal{O}_X(i\infty))|_{\operatorname{spec} A\times S} \\
&\cong \mathscr{E}|_{\operatorname{spec} A\times S} \otimes_{\mathcal{O}_{\operatorname{spec} A\times S}} \operatorname{pr}_{\operatorname{spec} A}^*(\mathcal{O}_X(i\infty)|_{\operatorname{spec} A}) \\
&\cong \mathscr{E}|_{\operatorname{spec} A\times S} \otimes_{\mathcal{O}_{\operatorname{spec} A\times S}} \operatorname{pr}_{\operatorname{spec} A}^* \mathcal{O}_{\operatorname{spec} A} \\
&\cong \mathscr{E}|_{\operatorname{spec} A\times S}.
\end{aligned}$$

Corollary 4.6
It is $\varinjlim \mathscr{M}_i \cong \mathscr{M}$.

Proof It is

$$\begin{aligned} \varinjlim \mathscr{M}_i &\cong \varinjlim \mathrm{pr}_{S*} \mathscr{E}_i \\ &\cong \mathrm{pr}_{S*} \varinjlim \mathscr{E}_i && \text{(proposition 2.67)} \\ &\cong \mathrm{pr}_{S*} \iota_* \mathscr{E}|_{\mathrm{spec}\, A \times S} && \text{(proposition 2.67)} \\ &\cong \mathrm{pr}_{S*} \mathscr{E}|_{\mathrm{spec}\, A \times S} \\ &\cong \mathscr{M}. \end{aligned}$$

We define below an A-module structure on the $\mathcal{O}_S$-module $\mathscr{M}$.

Remark 4.7
1) For all $i, j \in \mathbb{Z}$ the canonical isomorphisms

$$\mathcal{O}_X(i\infty) \otimes_{\mathcal{O}_X} \mathcal{O}_X(j\infty) \longrightarrow \mathcal{O}_X((i+j)\infty)$$

define isomorphisms

$$\mathrm{pr}_X^* \mathcal{O}_X(i\infty) \otimes_{\mathcal{O}_{X\times S}} \mathscr{E}_j \longrightarrow \mathscr{E}_{i+j}.$$

2) Let $a \in A$. For all $j \geq \deg(a)$ it is $a \in \mathcal{O}_X(j\infty)(X)$. In particular a induces by the maps of part 1) for all $j \geq \deg(a)$ the $\mathcal{O}_X$-linear injective maps

$$\mathcal{O}_X(i\infty) \xrightarrow{a\cdot} \mathcal{O}_X((i+j)\infty)$$

and for this $\mathcal{O}_{X\times S}$-linear injective maps

$$\mathscr{E}_i \xrightarrow{a\cdot} \mathscr{E}_{i+j}.$$

Using the funktor pr_{S*} induces injective maps

$$\mathscr{M}_i \cong \mathrm{pr}_{S*} \mathscr{E}_i \xrightarrow{a\cdot} \mathrm{pr}_{S*} \mathscr{E}_{i+j} \cong \mathscr{M}_{i+j}.$$

3) The maps of part 2) are compatible with the inductive system of the sheaves $\mathscr{M}_i$. They define $\mathcal{O}_S$-linear maps of the module $\mathscr{M}$.

4) As $\mathscr{M} \cong \mathrm{pr}_{S*} \mathscr{E}|_{\mathrm{spec}\, A\times S}$ there exists a canonical A-module structure on the $\mathcal{O}_S$-module $\mathscr{M}$. This structure coincides with the structure defined in part 3).

Lemma 4.8
The sheaf $\mathscr{M}$ is flat as an $\mathcal{O}_S$-module.

Proof The assertion is local in S ([Liu02], chapter 5, lemma 2.31, page 190). For this let $S = \operatorname{spec} R$ and define $M := H^0(S, \mathscr{M})$. We insert form the proof of corollary 4.6

$$M = H^0(S, \mathscr{M}) = H^0(\operatorname{spec} A \times S, \mathscr{E}|_{\operatorname{spec} A \times S}).$$

As a pullback of the $\mathcal{O}_{X\times S}$-vector bundle $\mathscr{E}$ the sheaf $\mathscr{E}|_{\operatorname{spec} A\times S}$ is an $\mathcal{O}_{\operatorname{spec} A\times S}$-vector bundle too. By assumption $\operatorname{spec} A \times S = \operatorname{spec} A \otimes R$ is an affine scheme. For this the $A \otimes R$-module $H^0(\operatorname{spec} A \times S, \mathscr{E}|_{\operatorname{spec} A\times S})$ is projective and in particular flat. The assertion follows now by lemma 2.6. □

Proposition 4.9
Let $\mathscr{E}$ be a vector bundle of general type over a field K, of rank $\frac{d}{\deg(\infty)}$ and of Euler characteristic $\chi(\mathscr{E}) = 0$. Let n_0 be maximal such that $H^0(X, \mathscr{E}_{n_0}) = 0$. Then $n_0 = 0$.

For proofing the proposition we first need the lemma below:

Lemma 4.10
It is $\chi(\mathscr{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\infty)) = \chi(\mathscr{E}) + d$.

Proof Cf. [LRS93], (2.9). □

Corollary 4.11
For all $i \in \mathbb{Z}$ it is $\chi(\mathscr{E}_i) = id$.

Proof (of proposition 4.9) We use for all $i \in \mathbb{Z}$ the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathscr{E}_i) \longrightarrow H^0(X, \mathscr{E}_{i+1}) \longrightarrow H^0(X, \mathscr{E}_{i+1}/\mathscr{E}_i) \longrightarrow \\ \longrightarrow H^1(X, \mathscr{E}_i) \longrightarrow H^1(X, \mathscr{E}_{i+1}) \longrightarrow 0. \end{aligned}$$

Because of the injectivity of the first map and of the surjectivity of the last map we get

$$h^0(X, \mathscr{E}_i) \le h^0(X, \mathscr{E}_{i+1}) \text{ and } h^1(X, \mathscr{E}_i) \ge h^1(X, \mathscr{E}_{i+1}).$$

By corollary 2.63 it is $h^0(X, \mathscr{E}_{i+1}/\mathscr{E}_i) = d$. From the sequence above we conclude

$$h^0(X, \mathscr{E}_i) - h^0(X, \mathscr{E}_{i+1}) + d \underbrace{-h^1(X, \mathscr{E}_i) + h^1(X, \mathscr{E}_{i+1})}_{\le 0} = 0.$$

This implies
$$h^0(X,\mathscr{E}_i) - h^0(X,\mathscr{E}_{i+1}) + d \geq 0$$
and finally
$$0 \leq h^0(X,\mathscr{E}_{i+1}) - h^0(X,\mathscr{E}_i) \leq d.$$
In particular by the choice of n_0, by lemma 4.10 and as $\mathscr{E}$ is of general type we conclude

$$h^0(X,\mathscr{E}_{n_0+1}) - h^0(X,\mathscr{E}_{n_0}) = (n_0+1)d \leq d \quad \Longrightarrow \quad n_0 \leq 0.$$

By $\chi(\mathscr{E}_{n_0+1}) = h^0(\mathscr{E}_{n_0+1}) = (n_0+1)d > 0$ we get $n_0 \geq 0$. This proofs the assumption. □

Remark 4.12
By proposition 4.9 we see that if $\chi(\mathscr{E}) = l$ for $l \in \mathbb{Z}$, then the calculation above implies
$$0 < l + (n_0+1)d \leq d$$
and we conclude $n_0 = \lceil \frac{-l}{d} \rceil$.

Corollary 4.13
As by definition it is $\mathscr{E}_0 = \mathscr{E}$ we get

$$H^0(X \times s, \mathscr{E}_s) = H^1(X \times s, \mathscr{E}_s)) = 0 \quad \forall s \in S.$$

Proposition 4.14
Let S be a local noetherian scheme. Then for all $i \in \mathbb{Z}$ the $\mathcal{O}_S$-modules $R^0 \operatorname{pr}_{S*} \mathscr{E}_i$ and $R^1 \operatorname{pr}_{S*} \mathscr{E}_i$ are projective.

Proof It is sufficient to proof the assertion for an affine scheme $S = \operatorname{spec} R$. By theorem 2.44 we conclude by

$$H^k(X \times s, \mathscr{E}_s) = 0 \quad \forall s \in S$$

that it is
$$H^k(X \times S, \mathscr{E}) = 0$$
for $k = 0, 1$. By the long sequence in cohomology

$$0 \longrightarrow H^0(X \times S, \mathscr{E}_i) \longrightarrow H^0(X \times S, \mathscr{E}_{i+1}) \longrightarrow H^0(X \times S, \mathscr{E}_{i+1}/\mathscr{E}_i) \longrightarrow$$
$$\longrightarrow H^1(X \times S, \mathscr{E}_i) \longrightarrow H^1(X \times S, \mathscr{E}_{i+1}) \longrightarrow 0$$

we conclude for $i = 0$

$$H^0(X,\mathscr{E}_1) \cong H^0(X,\mathscr{E}_1/\mathscr{E}_0).$$

By corollary 2.63 for all $i \in \mathbb{Z}$ the R-module $H^0(X, \mathscr{E}_{i+1}/\mathscr{E}_i)$ is projective of rank d.
For $i = -1$ we similar achieve

$$H^0(X, \mathscr{E}_0/\mathscr{E}_{-1}) \cong H^1(X, \mathscr{E}_{-1}).$$

If $i > 0$ then the sequence

$$0 \longrightarrow H^0(X, \mathscr{E}_i) \longrightarrow H^0(X, \mathscr{E}_{i+1}) \longrightarrow H^0(X, \mathscr{E}_{i+1}/\mathscr{E}_i) \longrightarrow 0$$

is split exact. As the direct sum of projective modules is again projective we get the assertion by induction.
Similarly for $i < 0$ the sequence

$$0 \longrightarrow H^0(X, \mathscr{E}_{i+1}/\mathscr{E}_i) \longrightarrow H^1(X, \mathscr{E}_i) \longrightarrow H^1(X, \mathscr{E}_{i+1}) \longrightarrow 0$$

is exact. The assertion again follows now by induction. □

Proposition 4.15
Let S be an arbitrary scheme. Then for all $i \in \mathbb{Z}$ the $\mathcal{O}_S$-modules $R^0 \operatorname{pr}_{S*} \mathscr{E}_i$ and $R^1 \operatorname{pr}_{S*} \mathscr{E}_i$ are projective.

Proof It is sufficient to proof the assertion for an affine scheme $S = \operatorname{spec} R$. By proposition 4.14 we can use the results of section 2.6. The assumptions of proposition 2.48 and proposition 2.49 are satisfied. In particular we can copy the proof of corollary 4.14 to the situation of an arbitrary ring R. □

Corollary 4.16
For all $i \in \mathbb{Z}$ the $\mathcal{O}_S$-modules $R^0 \operatorname{pr}_{S*} \mathscr{E}_i$ and $R^1 \operatorname{pr}_{S*} \mathscr{E}_i$ are vector bundles and it is

$$\operatorname{rk}(R^0 \operatorname{pr}_{S*} \mathscr{E}_i) = \begin{cases} id \text{ for } i \geq 0 \\ 0 \text{ for } i \leq 0 \end{cases}$$

and

$$\operatorname{rk}(R^1 \operatorname{pr}_{S*} \mathscr{E}_i) = \begin{cases} 0 \text{ for } i \geq 0 \\ -id \text{ for } i \leq 0. \end{cases}$$

Lemma 4.17
For all $i \in \mathbb{Z}$ and all $a \in A$, $a \neq 0$ the maps

$$\mathcal{O}_X(i\infty)/\mathcal{O}_X((i-1)\infty) \xrightarrow{a\cdot} \mathcal{O}_X((i+\deg(a))\infty)/\mathcal{O}_X((i-1+\deg(a))\infty)$$

are isomorphisms.

Proof We can proof the statement at the stalks of the closed points of X. Let $x \in X$ be a closed point. If $x \neq \infty$ it is $\mathcal{O}_X(i\infty)_x = \mathcal{O}_{X,x}$ and we have to show nothing. Let $\pi_\infty \in \mathcal{O}_{X,\infty}$ be a unifromizing element. Then there exists an element $\varepsilon \in \mathcal{O}^\star_{X,\infty}$ such that $a = \varepsilon\pi^{\infty(a)}$. Then it follows

$$\mathcal{O}_X(i\infty)_\infty = \pi_\infty^{-i}\mathcal{O}_{X,\infty}$$

and for all $i \in \mathbb{Z}$ the maps

$$\pi^{-i}\mathcal{O}_{X,\infty}/\pi^{-i+1}\mathcal{O}_{X,\infty} \xrightarrow{\varepsilon\pi^{\infty(a)}} \pi^{-i+\infty(a)}\mathcal{O}_{X,\infty}/\pi^{-i+1+\infty(a)}\mathcal{O}_{X,\infty}$$

are isomorphisms. □

Corollary 4.18
For all $i \geq 0$ and for all $a \in A$, $a \neq 0$ the maps

$$\operatorname{pr}_{S*}\mathscr{E}_{i+1}/\operatorname{pr}_{S*}\mathscr{E}_i \xrightarrow{a\cdot} \operatorname{pr}_{S*}\mathscr{E}_{i+1+\deg(a)}/\operatorname{pr}_{S*}\mathscr{E}_{i+\deg(a)}$$

are isomorphisms.

Proof By lemma 4.17 multiplication by a induces for all $i \in \mathbb{Z}$ isomorphisms

$$\mathscr{E}_{i+1}/\mathscr{E}_i \xrightarrow{a\cdot} \mathscr{E}_{i+1+\deg(a)}/\mathscr{E}_{i+\deg(a)}.$$

For all $i \geq 0$ it is $R^1\operatorname{pr}_{S*}\mathscr{E}_i = 0$ and we get

$$\operatorname{pr}_{S*}\mathscr{E}_{i+1}/\operatorname{pr}_{S*}\mathscr{E}_i = \operatorname{pr}_{S*}(\mathscr{E}_{i+1}/\mathscr{E}_i).$$

This proofs the statement. □

4.4 Category II $\Longrightarrow$ Category I

We first examine the affine case $S = \operatorname{spec} R$. The data of category II add in this case up to:
Let M be an $A \otimes R$-module equipped with an ascending filtration of R modules M_i such that:

a) It is $\varinjlim M_i = M$.

b) For all $i \in \mathbb{Z}$ the R modules M_i are projective of rank $\max(di, 0)$ and the modules M_{i+1}/M_i are projective of rank d for $i \geq 0$ and respectively of rank 0 for $i < 0$.

c) For all $a \in A$, $a \neq 0$ it is $aM_i \subseteq M_{i+\deg(a)}$ and the map

$$M_{i+1}/M_i \xrightarrow{a\cdot} M_{i+1+\deg(a)}/M_{i+\deg(a)}$$

is an isomorphism for all $i \geq 0$.

Remark 4.19
The data of category II are compatible with base change in R.

By proposition 2.36 and by the flat base change $S/\mathbb{F}_q$ it is

$$\begin{aligned}\operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0(X \times S, \mathcal{O}_{X\times S}(i\infty)\right) &\cong \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i\infty)) \otimes R\right)\\ &\cong X \times S.\end{aligned}$$

Let be $\mathcal{S}_X := \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i\infty))$. The module $\mathcal{M} := \bigoplus_{i=0}^{\infty} M_i$ is in a canonical way a graduate $\mathcal{S}_X \otimes R$ module. For this we can use the Proj functor and get a sheaf of $\mathcal{O}_{X\times S}$-modules $\mathscr{E}$. We show below that it satisfies the conditions of category I.

Proposition 4.20
The $\mathcal{O}_{X\times S}$-module $\mathcal{M}^\sim$ is a vector bundle of general type and of rank $d/\deg(\infty)$.

We will show first that in the case of a reduced noetherian ring R the $A\otimes R$-module $\mathcal{M}|_{\operatorname{spec} A\times S}$ is projective and finitely generated and we will calculate its rank (4.22 – 4.28). Then we proof this statement for $\mathcal{M}|_{\operatorname{spec} k(\infty)\times S}$ (4.29). Both statements and proposition 2.69 imply the statement of 4.20 in the case of a reduced noetherian ring R. Then we will proof in 4.31 the case of an arbitrary noetherian R, using theorem 2.8. Finally we construct in 4.32 for the general case an explicit descend data.

Lemma 4.21
The R-module $\mathcal{M}$ is projective.

Proof As $\mathcal{M} = \bigoplus_{i=0}^{\infty} M_i$ is a direct sum of projective R modules it is itself projective. □

Lemma 4.22
The $\mathcal{S}_X \otimes R$-module $\mathcal{M}$ is finitely generated.

Proof By assumption for all $i \geq 0$ the sequence

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0$$

splits and we conclude

$$M_{i+1} \cong \bigoplus_{k=0}^{i} M_{k+1}/M_k.$$

Let $0 \neq a \in H^0(X, \mathcal{O}_X(\deg(a)\infty) \subset A$. Then by assumption we get for all $i \geq 0$

$$M_{i+1}/M_i \cong M_{i+1+\deg a}/M_{i+\deg a}.$$

The R-modules $M_0, \ldots, M_{\deg a}$ thus generate $\mathcal{M}$ as an $\mathcal{S}_X \otimes R$-module. As the modules $M_0, \ldots, M_{\deg a}$ are by assumption finitely generated R-modules we get the assertion. □

We will now regard the restriction of $\mathcal{M}^{\sim}$ on spec $A \times S$.

Lemma 4.23
It is

$$(\mathcal{M})^{\sim}|_{\operatorname{spec} A \times \operatorname{spec} R} \cong M^{\sim}.$$

Proof In the setting of corollary 2.38 we consider the map

$$\begin{aligned} (\mathcal{M}[e^{-1}])_{(0)} &\longrightarrow M \\ \frac{m}{e^i} &\longmapsto m. \end{aligned}$$

It is an isomorphism. □

Lemma 4.24
The R-module M is projective.

Proof By assumption for all $i \geq 0$ the sequence

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0$$

split and we conclude

$$M_{i+1} \cong \bigoplus_{k=0}^{i} M_{k+1}/M_k.$$

Going to the limit we get

$$M \cong \bigoplus_{k=0}^{\infty} M_{k+1}/M_k.$$

As a direct sum of projective R-modules M is itself a projective R-module. □

Lemma 4.25
The $R \otimes A$-module M is finitely generated.

Proof Let $a \in A$ be such that $\deg a \neq 0$. By assumption it is for all $i \geq 0$

$$M_{i+1}/M_i \cong M_{i+1+\deg a}/M_{i+\deg a}.$$

Using the decomposition of lemma 4.24 we conclude that M as a $R \otimes A$ module is already generated by $M_1/M_0, \dots, M_{\deg a}/M_{\deg a-1}$. These modules are by assumption finitely generated R-modules. This proofs the lemma. □

Definition 4.26
We define

$$\begin{aligned} \deg : M &\longrightarrow \mathbb{Z} \\ m &\longmapsto \min\{i \mid m \notin M_{i-1}\}. \end{aligned}$$

Lemma 4.27
Let $0 \neq a \in A$. Then it is $\deg(am) = \deg a + \deg m$.

Proof Let be $\deg m = i$. Then it is $m \notin M_{i-1}$. As the map

$$M_i/M_{i-1} \xrightarrow{a} M_{i+\deg a}/M_{i-1+dega}$$

is by assumption an isomorphism we conclude $am \in M_{i+\deg a}$ and $am \notin M_{i+\deg a-1}$. This proofs the statement. □

Proposition 4.28
The module M is a projective $R \otimes A$-module of rank $\frac{d}{\deg(\infty)}$.

For proofing the proposition we copy the method of proposition 3.8.

4.4.1 $\mathbf{R = L}$ is a field and $\mathbf{A = \mathbb{F}_q[T]}$

Let $T \in \mathbb{F}_q[T]$. Then it is $\deg T = 1$. By lemma 4.25 the $L[T]$-module M is generated by a L-base of the L-vector space M_1. By assumption it has dimension d.

4.4.2 $\mathbf{R = L}$ is a field

The assumption follows on the lines of subsection 3.3.2.

4.4.3 R is a reduced noetherian ring

The assumption follows on the lines of subsection 3.3.3.

Proposition 4.29
$\mathcal{M}|_{\operatorname{spec} k(\infty)\times S}$ is a projective $k(\infty)\otimes R$-module of rank $\frac{d}{\deg(\infty)}$.

Proof Let $e := 1 \in H^0(X, \mathcal{O}_X(\infty))$. Then it is

$$\mathcal{M}|_{\operatorname{spec} k(\infty)\times S} \cong \mathcal{M}/e\mathcal{M} \cong \bigoplus_{i=0}^{\infty} M_{i+1}/M_i.$$

By corllary 2.39 we have

$$\operatorname{Proj}\left(\mathcal{S}_X \otimes R/e(\mathcal{S}_X\otimes R)\right) \cong \operatorname{spec} k(\infty)\otimes R.$$

If $a \in A$ such $\deg a > 0$ then

$$\operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} M_{i+1}/M_i\right) \cong \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} M_{1+i\deg a}/M_{i\deg a}\right).$$

For all $i \geq 0$ we have by assertion

$$M_1 \cong M_{1+\deg a}/M_{\deg a} \cong M_{1+i\deg a}/M_{i\deg a}$$

and we conclude

$$\bigoplus_{i=0}^{\infty}(M_{1+i\deg a}/M_{i\deg a}[a^{-1}])_{(0)} \cong M_1.$$

Thus we have to show that M_1 is a projective $k(\infty)\otimes R$-module of rank $\frac{d}{\deg(\infty)}$. As with R also $k(\infty)\otimes R$ is a reduced noetherian Ring and we can use proposition 2.7. By assertion M_1 is a projective R-module of rank d. Let $\mathfrak{p}$ be a prime Ideal of R. Define $K(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $M_1(\mathfrak{p}) := M_1 \otimes_R K(\mathfrak{p})$ Then $M_1(\mathfrak{p})$ is a $k(\infty)\otimes K(\mathfrak{p})$-module. Let $r_{\mathfrak{p}} \in \mathbb{N}$ be such that $\mathbb{F}_{q^{r_{\mathfrak{p}}}} := k(\infty)\cap K(\mathfrak{p})$. Then we have

$$k(\infty)\otimes K(\mathfrak{p}) \cong k(\infty)\otimes \mathbb{F}_{q^{r_{\mathfrak{p}}}} \otimes_{\mathbb{F}_{q^{r_{\mathfrak{p}}}}} K(\mathfrak{p}).$$

and $k(\infty)\otimes_{\mathbb{F}_{q^{r_{\mathfrak{p}}}}} K(\mathfrak{p})$ is a field. If $e \in k(\infty)\otimes \mathbb{F}_{q^{r_{\mathfrak{p}}}}$ is a non-trivial idempotent, then the idempotents $\{\sigma e\}$ for $\sigma \in \operatorname{Gal}(\mathbb{F}_{q^{r_{\mathfrak{p}}}}/\mathbb{F}_q)$ define a decomposition

$$k(\infty)\otimes \mathbb{F}_{q^{r_{\mathfrak{p}}}} \cong \prod_{\sigma\in\operatorname{Gal}(\mathbb{F}_{q^{r_{\mathfrak{p}}}}/\mathbb{F}_q)} k(\infty)$$

Thus the module $M_1(\mathfrak{p})$ decomposes in equal dimensional $K(\mathfrak{p})$-vector spaces $M_1(\mathfrak{p})_\sigma$ for $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^{r_\mathfrak{p}}}/\mathbb{F}_q)$ of dimension $\frac{d}{r_\mathfrak{p}}$. We conclude

$$\dim_{k(\infty)\otimes_{\mathbb{F}_{q^{r_\mathfrak{p}}}} K} M_1(\mathfrak{p})_\sigma = \frac{\dim_{K(\mathfrak{p})} M_1(\mathfrak{p})_\sigma}{\dim_{F_q^{r_\mathfrak{p}}} k(\infty)} = \frac{d}{\deg \infty}$$

This number is independent of $\mathfrak{p}$ and we are done. □

Corollary 4.30
Let R be a noetherian reduced ring. Then the $\mathcal{O}_{X\times S}$-module $\mathrm{Proj}\,\mathcal{M}$ *is locally free of rank* $\frac{d}{\deg \infty}$.

Proof The proof of the corollary follows form the stability of the dimension of the fibers. □

Corollary 4.31
Let R be a noetherian ring. Then the $\mathcal{O}_{X\times S}$-module $\mathrm{Proj}\,\mathcal{M}$ *is locally free of rank* $\frac{d}{\deg \infty}$.

Proof As $\mathcal{M}$ is a projective in particular flat R-module we can use theorem 2.8 from subsection 3.3.4 (cf. [Mat86], § 22, remark below theorem 22.6, page 177f). For this we define $I := \mathrm{Rad}(R)$. Then we get

$$\mathcal{M} \otimes_{\mathcal{S}_X \otimes R} \mathcal{S}_X \otimes I \cong \mathcal{M} \otimes_R I \cong I M.$$ □

4.4.4 R is an arbitrary ring

We want to lead back the general case by base change on the noetherian case. For this we need first the lemma below.

Lemma 4.32
Let R be a K-algebra and let M be a projective finitely generated R module of rank d. Then there exists a noetherian subring $R' \subseteq R$ and a projective finitely generated R' module $M' \subseteq M$ such that

$$M = R \otimes_{R'} M' \textit{ and } \mathrm{rang}\, M = \mathrm{rang}\, M'.$$

Proof Let M be generated by $m_1, \ldots, m_k \in M$. Then there exists a surjectiv map

$$\varphi : R^k \longrightarrow M, \quad e_i \longmapsto m_i.$$

As M is a projective R-module there exists a section $s : M \longrightarrow R^k$. For all $1 \le i \le k$ let $s(m_i) = (\lambda_{i1}, \ldots, \lambda_{ik}) \in R^k$ and let $R' := K[\lambda_{11}, \ldots, \lambda_{kk}]$.

As a finitely generated K-algebra R' is noetherian. Let M' be generated by $m_1 \ldots, m_k$ as a R'-submodule in M. We consider the surjectiv map

$$\varphi' : R'^k \longrightarrow M', \quad e_i \longmapsto m_i.$$

By construction the restriction of the section $s|_{M'}$ restricts to R'^k and is a section of φ'. In particular M' is a projective R'-module. By the diagram

$$\begin{array}{ccc} R^k = R \otimes_{R'} R'^k & \xrightarrow{R \otimes_{R'} \varphi'} & R \otimes_{R'} M' \\ \Big\| & & \Big\downarrow \\ R^k & \underset{s}{\overset{\varphi}{\rightleftarrows}} & M \end{array}$$

we can conclude a canonical isomorphism R-modules $R \otimes_{R'} M' \cong M$ of R-modules.
As M' is a finitely generated projective R'-module there exists an open covering of $\operatorname{spec} R' = \bigcup U_i$ such that $\operatorname{rang} M' = d'$ is constant. As R' is a subring of R the image of the map $\operatorname{spec} R \longrightarrow \operatorname{spec} R'$ is dense. In particular in any open subset U_i there exists a prime ideal $\mathfrak{p} \in \operatorname{spec} R'$ being an imagepoint of a prime ideal $\mathfrak{P} \in \operatorname{spec} R$. We conclude

$$R_{\mathfrak{P}} \otimes_{R'_{\mathfrak{p}}} M'_{\mathfrak{p}} \cong M_{\mathfrak{P}}$$

and $d' = \operatorname{rang} M$. □

By using lemma 4.32 we can reduce the statement of proposition 4.20 to the noetherian case:
By assumption there exists for all $i \geq 1$ an exact sequence

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0$$

of projective R modules M_i of rank di. The R-module M_{i+1}/M_i is projective of rank d for all $i \in \mathbb{Z}$. In particular the sequence splits and for all $i \in \mathbb{Z}$ we can assume that $N_{i+1} := M_{i+1}/M_i$ is a direct summand of the module M_{i+1}. By lemma 4.32 there exists for all $0 < i \leq \deg a$ a projective finitely generated R'-module N'_i such that $R \otimes_{R'} N'_i = N_i$. We define inductively the R' modules M'_i as follows. For $i = 1$ we set $M'_1 = N'_1$. For $i > 1$ let M'_i be the R' submodule M_i generated by N'_i and M'_{i-1}. We get for all i the exact sequence

$$0 \longrightarrow M'_i \longrightarrow M'_{i+1} \longrightarrow N'_{i+1} \longrightarrow 0.$$

We see by induction that the sequence splits and M_i' is a projective R'-module of rank id. Tensorizing the sequence by $R\otimes_{R'}$ implies

$$R \otimes_{R'} M_i' \cong M_i.$$

If $i > \deg a$ we define the R'-modules M_i' by using the maps given by multiplication with a. This construction produces a R'-module M' equipped with a filtration M_i' of category II. Using base change we conclude the assumption. Therewith proposition 4.20 is proofed.

Finally we show that the cohomology of $\mathcal{M}^{\sim}$ satisfies the conditions of a vector bundle of general type.
We adept the construction of section 3.4. First we enlarge the degree function on M up to $M \otimes_A F$.

$$\begin{aligned} \deg : M \otimes_A F &\longrightarrow \mathbb{Z} \\ m \otimes x &\longmapsto \deg m + \deg x \end{aligned}$$

By lemma 4.27 this is well defined.
At the fiber of ∞ we define the $\mathcal{O}_{X,\infty} \otimes R$-modules for all $i \in \mathbb{Z}$ by

$$M_{i,\infty} := \{\alpha \in M \otimes_A F \mid \deg \alpha \leq i\}.$$

Lemma 4.33
For all $i \geq 0$ it is

$$M_{i+1}/M_i \cong M_{i+1,\infty}/M_{i,\infty}.$$

Proof The canonical map $M_i \longrightarrow M_{i,\infty}$ is injective. It is $M_{i+1} \cap M_{i,\infty} = M_i$ and we get an injective R linear morphism

$$M_{i+1}/M_i \hookrightarrow M_{i+1,\infty}/M_{i,\infty}.$$

We have to show surjectivity. Let $m \otimes x \in M_{i+1,\infty} \setminus M_{i,\infty}$ be such that $x = \frac{a}{b}$ and such that $0 \neq a, b \in A$. Then $\deg m = i - \deg x$ and we get

$$am \in M_{i-degx+dega,\infty} = M_{i+\deg b,\infty}.$$

By assumption

$$M_i/M_{i-1} \xrightarrow{\ b\ } M_{i+\deg b}/M_{i+\deg b-1}$$

is an isomorphism so there is an element $\tilde{m} \in M_i$ such that

$$b\tilde{m} \equiv am \mod M_{i+\deg b-1}.$$

We conclude

$$\begin{aligned} & b\tilde{m} \otimes \frac{1}{b} - am \otimes \frac{1}{b} \equiv 0 \mod M_{i-1,\infty} \\ \Longrightarrow \quad & \tilde{m} \otimes 1 \equiv m \otimes \frac{a}{b} \mod M_{i-1,\infty}. \end{aligned}$$

This proofs the statement. □

Lemma 4.34
Let $\mathfrak{p}_\infty \subseteq \mathcal{S}_X$ be the homogeneous prime ideal corresponding to $\infty \in X$. Then for all $k \in \mathbb{Z}$ there exists a canonical $\mathcal{O}_{X,\infty} \otimes R$-linear isomorphism

$$(\mathcal{M}[k])_{(\mathfrak{p}_\infty)} \cong M_{k,\infty}.$$

Proof We consider the map

$$\begin{aligned} M \otimes_A F &\longrightarrow (\mathcal{M}[k])_{(\mathfrak{p}_\infty)} \\ m \otimes \frac{1}{f} &\longmapsto \frac{m}{f}. \end{aligned}$$

This maps induces the isomorphism of the lemma. □

Corollary 4.35
Let $\mathscr{E}_k := \operatorname{Proj}(\mathcal{M}[k])$. Then

$$\mathscr{E}_k|_{\operatorname{spec} \mathcal{O}_{X,\infty} \times S} \cong M_{k,\infty}^{\sim}.$$

Corollary 4.36
For all $i \in \mathbb{Z}$ it is

$$\mathscr{E}_i(X \times S) = \{m \in M \mid \deg m \leq i\}.$$

Corollary 4.37
It is:

1) $H^0(X \times S, \mathscr{E}_i) = H^0(X \times s, (\mathscr{E}_i)_s) = 0$ for $i \leq 0$ and $s \in S$.

2) $H^1(X \times S, \mathscr{E}_i) = H^1(X \times s, (\mathscr{E}_i)_s) = 0$ for $i \geq 0$ and $s \in S$.

3) $\varinjlim_{i \in \mathbb{Z}} H^0(X \times S, \mathscr{E}_i) = M$

Proof

1) Follows form corollary 4.36.

2) By lemma 2.61 the support of $\mathscr{E}_i/\mathscr{E}_{i-1}$ is contained in $\operatorname{spec}\mathcal{O}_{X,\infty}\otimes S$. By lemma 4.33 for all $i \geq 0$

$$M_{i+1}/M_i \cong M_{i+1,\infty}/M_{i,\infty}$$

is an isomorphism of R-modules. By assumption M_{i+1}/M_i is finitely generated and for this $M_{i+1,\infty}/M_{i,\infty}$ is generated by global sections. We conclude that the canonical map

$$H^0(X\times S,\mathscr{E}_{i+1}) \longrightarrow H^0(X\times S,\mathscr{E}_{i+1}/\mathscr{E}_i)$$

is surjectiv for all $i \geq 0$. In particular it is $H^1(X\times S,\mathscr{E}_i)=0$ for all $i\geq 0$.

3) Follows form corollary 4.36. □

Remark 4.38
If we replace the condition of proposition 4.3 more general by $\chi(\mathscr{E}) = l$ for $l \in \mathbb{Z}$ then we have to replace the condition of category II by $\operatorname{rk}(M_i) = \max(l+id,0)$ and for all $0 \neq a \in A$

$$M_i/M_{i-1} \xrightarrow{\ a\ } M_{i+\deg a}/M_{i-1+\deg a}$$

is an isomorphism for all non vanishing quotients M_i/M_{i-1}.

Remark 4.39
If S is an arbitrary scheme then

$$\bigoplus_{i=0}^{\infty}\mathscr{M}_i$$

is a graduate $\mathcal{S}_X\otimes\mathcal{O}_S$-module. Using the Proj construction we can define a $\mathcal{O}_{X\times S}$-module $\mathscr{E}$. As the data of category I could be checked on an open affine covering of S we conclude by the previous results that $\mathscr{E}$ is an $\mathcal{O}_{X\times S}$ vector bundle of general type.

4.5 Functorial equivalence of the categories

We consider category I as a subcategory of the category of quasi coherent $\mathcal{O}_{X\times S}$-modules and consider category II as a subcategory of the category of graduate $\mathcal{S}_{X\times S}$-modules.
The construction of section 4.3 is done by using the functor $\Gamma_*(\cdot)$. Thus it is functorial. The construction of section 4.4 is done by using the functor $(\cdot)^\sim$. So it is functorial too.

By [GD61], chapter II, 3.3, page 56ff, there exists natural transformations

$$\alpha : (\cdot) \longrightarrow \Gamma_*((\cdot)^\sim)$$
$$\beta : (\Gamma_*(\cdot))^\sim \longrightarrow (\cdot)$$

for the compositions of the functors $\Gamma_*(\cdot)$ and $(\cdot)^\sim$.

Lemma 4.40
The natural transformation β induces an isomorphism on the objects of category I.

Proof The statement is local in S. Thus we can assume that S is an affine scheme. By proposition 2.33 β induces for all objects of category I an isomorphism. This proofs the statement. □

Lemma 4.41
The natural transformation α induces an isomorphism on the objects of category II.

Proof The statement is local in S. Thus we can assume that S is an affine scheme. Let (M_i, M) be an object of category II and define $\mathcal{M} := \bigoplus_{i\geq 0} M_i$. By [Har77], chapter II, exercise 5.9, page 125 α induces isomorphisms

$$M_i \cong (\Gamma_*(\mathcal{M}^\sim))_i$$

for all $i \geq n_0$ and an appropriate number $n_0 \in \mathbb{N}$.
Let $a \in A$ be such that $\deg a > 0$. By assumption

$$M_{i+1}/M_i \xrightarrow{\ a\ } M_{i+1+\deg a}/M_{i+\deg a}$$

is an isomorphism for all $i \geq 0$. We get

$$\begin{aligned} M_{i+1}/M_i &\cong M_{i+1+\deg a^{n_0}}/M_{i+\deg a^{n_0}} \\ &\cong (\Gamma_*(\mathcal{M}^\sim))_{i+1+\deg a^{n_0}}/(\Gamma_*(\mathcal{M}^\sim))_{i+\deg a^{n_0}} \\ &\cong (\Gamma_*(\mathcal{M}^\sim))_{i+1}/(\Gamma_*(\mathcal{M}^\sim))_i \end{aligned}$$

for all $i \geq 0$. The statement now follows by induction on $i \geq 0$. □

This completes the proof of theorem 4.3.

4.6 Drinfeld's theorem, extended version

By using remark 4.38 we can conclude the following extension of theorem 4.3.

Theorem 4.42 (Drinfeld, [Dri86])
The two categories above are equivalent:

I') Category of ascending families of $\mathcal{O}_{X\times S}$-vector bundles

$$\dashrightarrow \mathscr{E}_i \hookrightarrow \mathscr{E}_{i+1} \hookrightarrow \mathscr{E}_{i+2} \dashrightarrow$$

of rank $\frac{d}{\deg(\infty)}$ such that:

1) For all $i \in \mathbb{Z}$ it is $\mathscr{E}_i(\infty) \cong \mathscr{E}_{i+d}$.

2) For all $i \in \mathbb{Z}$ it is $\mathrm{pr}_{S*}\mathscr{E}_{i+1}/\mathscr{E}_i$ an $\mathcal{O}_S$ line bundle.

3) It is $H^0(X \times S, \mathscr{E}_{-1}) = H^1(X \times S, \mathscr{E}_{-1}) = 0$.

II') Category of $\mathcal{O}_S$-modules $\mathscr{M}$ equipped with an ascending filtration of $\mathcal{O}_S$-modules $\mathscr{M}_i$ and a ring homomorphism $A \longrightarrow \mathrm{End}_{\mathcal{O}_S}(\mathscr{M})$ such that

a) It is $\varinjlim \mathscr{M}_i = \mathscr{M}$.

b) For all $i \in \mathbb{Z}$ the $\mathcal{O}_S$-modules $\mathscr{M}_i$ are vector bundles of rank $\max(i+1, 0)$. The modules $\mathscr{M}_i/\mathscr{M}_{i-1}$ are vector bundles of rank 1 for $i \geq 0$, of rank 0 for $i < 0$.

c) For all $a \in A$, $a \neq 0$ it is $a\mathscr{M}_i \subseteq \mathscr{M}_{i+d\deg(a)}$ and the map

$$\mathscr{M}_i/\mathscr{M}_{i-1} \xrightarrow{a\cdot} \mathscr{M}_{i+d\deg(a)}/\mathscr{M}_{i-1+d\deg(a)}$$

is an isomorphism for all $i \geq 0$.

Proposition 4.43
Let $\mathscr{E}_i$ be as in theorem 4.42. Then for all $i \in \mathbb{Z}$ the $\mathcal{O}_{X\times S}$-vector bundle $\mathscr{E}_i$ is of general type and for all $s \in S$ it is $\chi((\mathscr{E}_i)_s) = i + 1$.

Proof As $H^1(X \times S, \mathscr{E}_{-1}) = 0$ the assertion of corollary 2.50 are satisfied. For this the cohomology of $\mathscr{E}_{-1}$ is compatible with arbitrary base change in S. In particular it is

$$0 = H^0(X \times S, \mathscr{E}_{-1}) \otimes_{\mathcal{O}_S} k(s) \cong H^0(X \times s, (\mathscr{E}_{-1})_s)$$

and

$$0 = H^1(X \times S, \mathscr{E}_{-1}) \otimes_{\mathcal{O}_S} k(s) \cong H^1(X \times s, (\mathscr{E}_{-1})_s)$$

for all $s \in S$. By lemma 4.2, 2) the sheaf $\mathscr{E}_{-1}$ is a vector bundle of general type.
For all $k \in \mathbb{Z}$ we have by lemma 4.2, 1) that $\mathscr{E}_{-1}(k\infty) = \mathscr{E}_{-1+kd}$ is of general type too. Thus it is sufficient to proof the statement for $0 \leq i < d-1$. If $k \geq 0$ then by the sequence

$$0 \longrightarrow \mathscr{E}_{-1+kd} \longrightarrow \mathscr{E}_{i+kd} \longrightarrow \mathscr{E}_{i+kd}/\mathscr{E}_{-1+kd} \longrightarrow 0$$

we conclude that $H^1(X \times s, (\mathscr{E}_{i+kd})_s) = 0$ for all $s \in S$. If $k < 0$ then by the sequence

$$0 \longrightarrow \mathscr{E}_{i+kd} \longrightarrow \mathscr{E}_{-1+(k-1)d} \longrightarrow \mathscr{E}_{-1+(k-1)d}/\mathscr{E}_{i+kd} \longrightarrow 0$$

we conclude $H^0(X \times s, (\mathscr{E}_{i+kd})_s) = 0$ for all $s \in S$. Thus by definition $\mathscr{E}_i$ is of general type.
The calculation of the Euler characteristic follows from

$$\chi((\mathscr{E}_{-1})_s) = 0 \text{ and } \chi((\mathscr{E}_i/\mathscr{E}_{i-1})_s) = 1$$

and the sequence

$$0 \longrightarrow (\mathscr{E}_{i-1})_s \longrightarrow (\mathscr{E}_i)_s \longrightarrow (\mathscr{E}_i/\mathscr{E}_{i-1})_s \longrightarrow 0.$$

□

Proof (of theorem 4.42)

Category I' $\Longrightarrow$ Category II' By proposition 4.43 for all $i \in \mathbb{Z}$ the $\mathcal{O}_{X\times S}$-vector bundles $\mathscr{E}_i$ are of general type and define by proposition 4.3 and remark 4.38 $\mathcal{O}_S$-vector bundles $\mathscr{M}_i := \operatorname{pr}_{S*} \mathscr{E}_i$. By assumption $\operatorname{pr}_{S*}(\mathscr{E}_i/\mathscr{E}_{i-1})$ is an $\mathcal{O}_S$-line bundle. If $i \geq 0$ then

$$\mathscr{M}_i/\mathscr{M}_{i-1} \cong \operatorname{pr}_{S*} \mathscr{E}_i / \operatorname{pr}_{S*} \mathscr{E}_{i-1} \cong \operatorname{pr}_{S*}(\mathscr{E}_i/\mathscr{E}_{i-1})$$

and if $i < 0$ then

$$\mathscr{M}_i \cong \operatorname{pr}_{S*} \mathscr{E}_i = 0.$$

Using theorem 4.3 we get the assumption.

Category II' $\Longrightarrow$ Category I' For all l such that $-1 \leq l < d-1$ the $\mathcal{O}_S$-vector bundles $(\mathscr{M}_{l+di})$ are objects of category II. Using theorem 4.3 we get $\mathcal{O}_{X\times S}$-vector bundles $\mathscr{E}_l$ of general type. For all $i \in \mathbb{Z}$ we define by

$$\mathscr{E}_{l+i} := \mathscr{E}_l(i\infty)$$

an ascending family of $\mathcal{O}_{X\times S}$ vector bundles. They satisfy by construction the condition 1). Because of this property it is sufficient to proof condition 2) for $i \geq 0$. In this case we have

$$\mathrm{pr}_{S*}\left(\mathscr{E}_i/\mathscr{E}_{i-1}\right) \cong \mathscr{M}_i/\mathscr{M}_{i-1}.$$

This implies condition 2).

For all $s \in S$ it is $\chi((\mathscr{E}_{-1})_s) = 0$. By corollary 4.13 and theorem 2.44 it is $H^0(X \times S, \mathscr{E}_{-1}) = H^1(X \times S, \mathscr{E}_{-1}) = 0$. This proofs the validity of condition 3).

Both constructions are functorial and the equivalence of the categories follows on the lines of section 4.5. □

5 Elliptic Sheaves

5.1 Definition of elliptic sheaves

Let X, S, ∞ be as in the previous section.

Definition 5.1
An elliptic sheaf $(\mathscr{E}_i, s_i, t_i)_{i\in\mathbb{Z}}$ consists of the following data:

1) $\mathscr{E}_i$: Vector bundles of rank d on $X \times S$

2) $s_i : \mathscr{E}_i \longrightarrow \mathscr{E}_{i+1}$: Injective morphisms of $\mathcal{O}_{X\times S}$-modules

3) $t_i : {}^\tau\mathscr{E}_i \longrightarrow \mathscr{E}_{i+1}$: Injective morphisms of $\mathcal{O}_{X\times S}$-modules

such that:

a) The diagram

$$\begin{array}{ccccccc} \dashrightarrow & \mathscr{E}_{i-1} & \xrightarrow{s_{i-1}} & \mathscr{E}_i & \xrightarrow{s_i} & \mathscr{E}_{i+1} & \dashrightarrow \\ & & \nearrow_{t_{i-1}} & & \nearrow_{t_i} & & \\ \dashrightarrow & {}^\tau\mathscr{E}_{i-1} & \xrightarrow{{}^\tau s_{i-1}} & {}^\tau\mathscr{E}_i & \xrightarrow{{}^\tau s_i} & {}^\tau\mathscr{E}_{i+1} & \dashrightarrow \end{array}$$

commutes.

b) It is $\mathscr{E}_{i+d\deg\infty} = \mathscr{E}_i \otimes_{\mathcal{O}_{X\times S}} \mathrm{pr}_X^* \mathcal{O}_X(\infty)$.

c) The diagram

$$\begin{array}{ccccccc} \mathscr{E}_i & \xrightarrow{s_i} & \mathscr{E}_{i+1} & \xrightarrow{s_{i+1}} & \mathscr{E}_{i+2} & \dashrightarrow & \mathscr{E}_{i+d\deg(\infty)} \\ & \searrow^{\text{canonical}} & & & & & \| \\ & & & & & & \mathscr{E}_i \otimes_{\mathcal{O}_{X\times S}} \mathrm{pr}_X^* \mathcal{O}_X(\infty) \end{array}$$

commutes.

d) The $\mathcal{O}_S$-module $(\mathrm{pr}_S)_*\mathscr{E}_i/\mathscr{E}_{i-1}$ is a line bundle on S.

e) The $\mathcal{O}_S$-module $(\mathrm{pr}_S)_*\mathscr{E}_i/{}^\tau\mathscr{E}_{i-1}$ is a line bundle on S.

5.2 Properties

For $i \in \mathbb{Z}$ let $(\mathscr{E}_i, s_i, t_i)$ be the data of an elliptic sheaf.

Remark 5.2
Equivalent are:

- The maps $t_i : {}^\tau\mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} \longrightarrow \mathscr{E}_{i+1}/\mathscr{E}_i$ are isomorphisms for all $i \in \mathbb{Z}$.
- For all $x \in X \times S$ and $i \in \mathbb{Z}$ it is
  1) ${}^\tau\mathscr{E}_{i,x} \cap \mathscr{E}_{i,x} = {}^\tau\mathscr{E}_{i-1,x}$
  2) ${}^\tau\mathscr{E}_{i,x} + \mathscr{E}_{i,x} = \mathscr{E}_{i+1,x}$.

Corollary 5.3
If one of the equivalent conditions of remark 5.2 is satisfied then it is

$$\mathscr{E}_{i,x}/{}^\tau\mathscr{E}_{i-1,x} \cong \mathscr{E}_{i+1,x}/{}^\tau\mathscr{E}_{i,x}$$

for all $i \in \mathbb{Z}$.

Proof It is

$$\begin{aligned}\mathscr{E}_{i+1,x}/{}^\tau\mathscr{E}_{i,x} &\cong ({}^\tau\mathscr{E}_{i,x} + \mathscr{E}_{i,x})/{}^\tau\mathscr{E}_{i,x} \\ &\cong \mathscr{E}_{i,x}/\mathscr{E}_{i,x} \cap {}^\tau\mathscr{E}_{i,x} \\ &\cong \mathscr{E}_{i,x}/{}^\tau\mathscr{E}_{i-1,x}.\end{aligned}$$

Corollary 5.4
In the setting of corollary 5.3 for all $i \in \mathbb{Z}$ the sequence

$$0 \longrightarrow {}^\tau\mathscr{E}_{i-1} \xrightarrow{(t_{i-1}, -{}^\tau s_{i-1})} \mathscr{E}_i \oplus {}^\tau\mathscr{E}_i \xrightarrow{(s_i + t_i)} \mathscr{E}_{i+1} \longrightarrow 0$$

is exact.

Proposition 5.5
Equivalent are:

a) For all $i \in \mathbb{Z}$ it is $\operatorname{supp} \mathscr{E}_i/\mathscr{E}_{i-1} \cap \operatorname{supp} \mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} = \emptyset$.

b) For all $i \in \mathbb{Z}$ the maps $t_i : {}^\tau\mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} \longrightarrow \mathscr{E}_{i+1}/\mathscr{E}_i$ are isomorphisms.

We show first that a) implies b).

Lemma 5.6
For all $i \in \mathbb{Z}$ let $\operatorname{supp} \mathscr{E}_i/\mathscr{E}_{i-1} \cap \operatorname{supp} \mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} = \emptyset$. Then the maps $t_i : {}^\tau\mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} \longrightarrow \mathscr{E}_{i+1}/\mathscr{E}_i$ are isomorphisms for all $i \in \mathbb{Z}$.

Proof By assumption for all $x \in X \times S$ either $(s_i)_x$ or $(t_i)_x$ is an isomorphism. □

By lemma 2.61 we have $\operatorname{supp} \mathcal{E}_i/\mathcal{E}_{i-1} \subseteq \operatorname{spec} k(\infty) \times S$ for all $i \in \mathbb{Z}$ and we can conclude the assertion „b) implies a)“ by the lemma below.

Lemma 5.7
For all $i \in \mathbb{Z}$ it is $\operatorname{supp}(\mathcal{E}_{i+1}/{}^\tau\mathcal{E}_i) \subseteq \operatorname{spec} A \times S$.

Proof The assertion is local in S. For this let $S = \operatorname{spec} R$ be an affine scheme. Let $x \in \operatorname{spec} k(\infty) \otimes R$ and let

$$\iota : \operatorname{spec} k(\infty) \otimes R \longrightarrow \operatorname{spec} \mathcal{O}_{X,\infty} \otimes R$$

be the canonical embedding. Let $\mathfrak{p} \subseteq \mathcal{O}_{X,\infty} \otimes R$ be the prime ideal associated to $\iota(x)$ such that $\mathfrak{p} \supseteq \mathfrak{m}_{X,\infty} \otimes R$. For all $i \in \mathbb{Z}$ it is

$$\begin{aligned}\mathcal{E}_i &= {}^\tau\mathcal{E}_{i-1} + \mathcal{E}_{i-1} \\ &= {}^\tau\mathcal{E}_{i-1} + {}^\tau\mathcal{E}_{i-2} + \mathcal{E}_{i-2} = {}^\tau\mathcal{E}_{i-1} + \mathcal{E}_{i-2} \\ &= \ldots \\ &= {}^\tau\mathcal{E}_{i-1} + \mathcal{E}_{i-d} = {}^\tau\mathcal{E}_{i-1} + \mathcal{E}_i \otimes_{\mathcal{O}_{X\times S}} \mathcal{O}_{X\times S}(-\infty).\end{aligned}$$

If we localize at the point x we get

$$\mathcal{E}_{i,x} = {}^\tau\mathcal{E}_{i-1,x} + \mathcal{E}_{i,x} \otimes_{\mathcal{O}_{X\times S,x}} \mathcal{O}_{X\times S}(-\infty)_x.$$

By definition of $\mathcal{O}_{X\times S}(-\infty)$ it is

$$\mathcal{O}_{X\times S}(-\infty)_x = (\mathfrak{m}_{X,\infty} \otimes R)_{\mathfrak{p}}.$$

We conclude

$$\mathcal{E}_{i,x} = {}^\tau\mathcal{E}_{i-1,x} + \mathcal{E}_{i,x} \otimes_{(\mathcal{O}_{X,\infty}\otimes R)_{\mathfrak{p}}} (\mathfrak{m}_{X,\infty} \otimes R)_{\mathfrak{p}}.$$

By assumption $\mathcal{E}_{i,x}$ is a finitely generated, projective $(\mathcal{O}_{X,\infty} \otimes R)_{\mathfrak{p}}$-module. We get

$$\mathcal{E}_{i,x} = {}^\tau\mathcal{E}_{i-1,x} + \mathcal{E}_{i,x}(\mathfrak{m}_{X,\infty} \otimes R)_{\mathfrak{p}}.$$

It is $(\mathfrak{m}_{X,\infty} \otimes R)_{\mathfrak{p}} \subseteq \operatorname{Rad}((\mathcal{O}_{X,\infty} \otimes R)_{\mathfrak{p}})$ and by the lemma of Nakayama we conclude $\mathcal{E}_{i,x} = {}^\tau\mathcal{E}_{i-1,x}$ for all $i \in \mathbb{Z}$. This proofs the assumption. □

Corollary 5.8
If one of the equivalent conditions of proposition 5.5 is satisfied then for all $i \in \mathbb{Z}$ it is

$$\operatorname{pr}_{S*} \mathcal{E}_i/{}^\tau\mathcal{E}_{i-1} = \operatorname{pr}_{S*} \left(\mathcal{E}_i/{}^\tau\mathcal{E}_{i-1}|_{\operatorname{spec} A\times S}\right).$$

5.3 Drinfeld modules and elliptic sheaves

Definition 5.9
Let $\mathscr{E}\ell\ell_X^{(d)}(S)$ be the category of elliptic sheaves of rank d such that additionally:

1) It is $\chi(\mathscr{E}_{-1,s}) = 0$ for all $s \in S$.

2) The maps $t_i : {}^\tau\mathscr{E}_i / {}^\tau\mathscr{E}_{i-1} \longrightarrow \mathscr{E}_{i+1}/\mathscr{E}_i$ are isomorphisms for all $i \in \mathbb{Z}$.

Theorem 5.10
The following categories are equivalent:

$$\left\{ \begin{array}{c} \textit{Standard Drinfeld modules} \\ \textit{of rank } d \textit{ over } S \end{array} \right\} \quad \leftrightarrow \quad \left\{ \mathscr{E}\ell\ell_X^{(d)}(S) \right\}$$

Below we will proof theorem 5.10.

5.3.1 Elliptic sheaves $\Longrightarrow$ Drinfeld modules

Remark 5.11
Let $S = \operatorname{spec} L$ be a field. Then the base change $\operatorname{Frob} : L \longrightarrow L$ is flat and it is $H^i(X \times S, {}^\tau\mathscr{E}) \cong L \otimes_L H^i(X \times S, \mathscr{E})$ and $h^i(\mathscr{E}) = h^i({}^\tau\mathscr{E})$ for $i = 0, 1$.

Lemma 5.12
Let $(\mathscr{E}_i, s_i, t_i)$ be an object of the category $\mathscr{E}\ell\ell_X^{(d)}(S)$ and let $S = \operatorname{spec} L$ be a field. Let $n_0 \in \mathbb{Z}$ be maximal such that $h^0(\mathscr{E}_{n_0}) = 0$. Then $n_0 = -1$.

Proof We first show that it is $H^1(X \times S, {}^\tau\mathscr{E}_{n_0+k-1}) = 0$ for all $k \geq 1$. It is sufficient to proof the assertion for $k = 1$, because we have $h^1(\mathscr{E}_j) \geq h^1(\mathscr{E}_i)$ for all $i \geq j$. Let

$$\begin{aligned} \mathscr{E}_{n_0+1}(X \times S) &\hookrightarrow {}^\tau\mathscr{E}_{n_0+1}(X \times S) \\ v &\longmapsto {}^\tau v := v \otimes 1 \end{aligned}$$

be for all $i \in \mathbb{Z}$ the canonical map. Let $0 \neq v \in \mathscr{E}_{n_0+1}(X \times S)$ and define $\tau v := t({}^\tau v)$.
Assumption: $\tau v \in \mathscr{E}_{n_0+1}(X \times S)$. Then we have

$$\tau v \in \mathscr{E}_{n_0+1}(X \times S) \cap {}^\tau\mathscr{E}_{n_0+1}(X \times S) = {}^\tau\mathscr{E}_{n_0}(X \times S) = 0$$

contrary to $\tau v \neq 0$. As we have $h^0(\mathscr{E}_{n_0+2}) \leq 2$ the map

$$\mathscr{E}_{n_0+1}(X \times S) \oplus {}^\tau\mathscr{E}_{n_0+1}(X \times S) \longrightarrow \mathscr{E}_{n_0+2}(X \times S)$$

is surjectiv and it follows that $H^1(X \times S, {}^\tau\mathscr{E}_{n_0}) = 0$ (cf. corollary 5.4).
By the exact sequence

$$0 \longrightarrow \mathscr{E}_{i-1} \longrightarrow \mathscr{E}_i \longrightarrow \mathscr{E}_i/\mathscr{E}_{i-1} \longrightarrow 0$$

we conclude $0 \le h^0(\mathscr{E}_{i+1}) - h^0(\mathscr{E}_i) \le 1$ for all $i \in \mathbb{Z}$ (cf. proof of proposition 4.9). We show below by induction that it is $h^0(\mathscr{E}_{n_0+k}) = k$. For $k = 0, 1$ this is the definition of n_0.
We consider for $k \ge 1$ the exact sequence of corollary 5.4

$$0 \longrightarrow {}^\tau\mathscr{E}_{n_0+k-1} \longrightarrow \mathscr{E}_{n_0+k} \oplus {}^\tau\mathscr{E}_{n_0+k} \longrightarrow \mathscr{E}_{n_0+k+1} \longrightarrow 0. \qquad \square$$

Using global sections and using $H^1(X \times S, {}^\tau\mathscr{E}_{n_0+k-1}) = 0$ we get the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X \times S, {}^\tau\mathscr{E}_{n_0+k-1}) \longrightarrow \\ \longrightarrow H^0(X \times S, \mathscr{E}_{n_0+k}(X \times S) \oplus H^0(X \times S, {}^\tau\mathscr{E}_{n_0+k}) \longrightarrow \\ \longrightarrow H^0(X \times S, \mathscr{E}_{n_0+k+1}) \longrightarrow 0 \end{aligned}$$

for all $k \ge 1$. It follows

$$-h^0({}^\tau\mathscr{E}_{n_0+k-1}) + h^0({}^\tau\mathscr{E}_{n_0+k}) + h^0(\mathscr{E}_{n_0+k}) = h^0(\mathscr{E}_{n_0+k+1}).$$

Thus

$$-(k-1) + 2k = k + 1 = h^0(\mathscr{E}_{n_0+k+1}).$$

On the other hand we have $\chi(\mathscr{E}_{n_0+k}) = h^0(\mathscr{E}_{n_0+k}) = n_0 + k + 1$ for $k \ge 0$. Thus for $k \ge 1$ we get altogether

$$k + 1 = h^0(\mathscr{E}_{n_0+k+1}) = n_0 + k + 2 \implies n_0 = -1.$$

Corollary 5.13
In the setting of lemma 5.12 let $0 \ne m_0 \in H^0(X \times S, \mathscr{E}_0)$. Then

$$m_0, \tau m_0, \ldots, \tau^i m_0$$

is a L-base of $H^0(X \times S, \mathscr{E}_i)$ for all $i \ge 0$.

Proof We show by induction on i: For all $i \ge 0$ it is $\tau^i m_0 \notin H^0(X \times S, \mathscr{E}_{i-1})$. For $i = 0$ it is $H^0(X \times S, \mathscr{E}_{-1}) = 0$. Let be $i > 0$. Assumption: $\tau^i m_0 \in H^0(X \times S, \mathscr{E}_{i-1})$. Then

$$\begin{aligned} \tau^i m_0 \in H^0(X \times S, \mathscr{E}_{i-1}) \cap H^0(X \times S, {}^\tau\mathscr{E}_{i-1}) \\ \implies {}^\tau(\tau^{i-1} m_0) \in H^0(X \times S, {}^\tau\mathscr{E}_{i-2}). \end{aligned}$$

On the other hand the map

$$H^0(X \times S, \mathcal{E}_{i-1})/H^0(X \times S, \mathcal{E}_{i-2}) \longrightarrow H^0(X \times S, {}^\tau\mathcal{E}_{i-1})/H^0(X \times S, {}^\tau\mathcal{E}_{i-2})$$

is injective thus we have $\tau^{i-1}m_0 \in H^0(X \times S, \mathcal{E}_{i-2})$ in contradiction to the induction hypothesis □

Proposition 5.14
Let $(\mathcal{E}_i, s_i, t_i)$ be an object of the category $\mathcal{E}\ell\ell_X^{(d)}(S)$. Then $\mathcal{E}_{-1}$ is a vector bundle of general type.

Proof By lemma 5.12 we conclude

$$H^0(X \times s, (\mathcal{E}_{-1})_s) = 0 \text{ and } H^1(X \times s, (\mathcal{E}_{-1})_s) = 0$$

for all $s \in S$. The assumption follows now form 4.2, 2). □

Proposition 5.15
Let $(\mathcal{E}_i, s_i, t_i)$ be an object of the category $\mathcal{E}\ell\ell_X^{(d)}(S)$. Then $(\mathcal{E}_i, s_i)$ is an object of category I' (cf. 4.42).

Proof The conditions 1) and 2) are satisfied by definition of an elliptic sheaf. Condition 3) follows by proposition 5.15, theorem 2.44, proposition 2.48, proposition 2.49 and proposition 4.15. □

Corollary 5.16
For all $i \in \mathbb{Z}$ it is

$$\mathrm{pr}_{S*}\,\mathcal{E}_i/{}^\tau\mathcal{E}_{i-1} \cong \mathrm{pr}_{S*}\,\mathcal{E}_0/{}^\tau\mathcal{E}_{-1} \cong \mathrm{pr}_{S*}\,\mathcal{E}_0.$$

Corollary 5.17
In the setting of corollary 5.16 we get by using the $\mathcal{O}_X$-module structure on $\mathcal{E}_i$ and corollary 5.16 an A-module structure on $\mathrm{pr}_{S*}\,\mathcal{E}_0$. Further on we get a ring homomorphism

$$A \longrightarrow \mathrm{End}_{\mathcal{O}_S}(\mathrm{pr}_{S*}\,\mathcal{E}_0) \cong \mathcal{O}_S(S).$$

Definition 5.18
Let $(\mathcal{E}_i, s_i, t_i)$ be an object of the category $\mathcal{E}\ell\ell_X^{(d)}(S)$. We call the map

$$\mathrm{char} : A \longrightarrow \mathrm{End}_{\mathcal{O}_S}(\mathrm{pr}_{S*}\,\mathcal{E}_0) \cong \mathcal{O}_S(S)$$

in corollary 5.17 the characteristic *of an elliptic sheaf.*

Corollary 5.19
Let $S = \operatorname{spec} R$ be an affine scheme. If $\mathrm{pr}_{S*}\mathscr{E}_0$ is a free R-module then for all $i \geq 0$ the R-modules $\mathrm{pr}_{S*}\mathscr{E}_i$ are free.

Proof Let $m_0 \in H^0(X \times S, \mathscr{E}_0)$ be a non vanishing section that is $0 \neq (m_0)_s \in H^0(X \times s, (\mathscr{E}_0)_s)$ for all $s \in S$. Then the elements

$$(m_0)_s, \tau(m_0)_s, \ldots \tau^i(m_0)_s \in H^0(X \times s, (\mathscr{E}_i)_s)$$

form a basis of $H^0(X \times s, (\mathscr{E}_i)_s)$ for all $s \in S$ by corollary 5.13. In particular they define a trivialization of the R-modules $\mathrm{pr}_{S*}\mathscr{E}_i$ for all $i \geq 0$. □

Corollary 5.20
In the setting of corollary 5.19 it is $H^0(U \times S, \mathscr{E}_0)$ a free $R\{\tau\}$-module of rank 1.

Proof By 2.67 it is $H^0(U \times S, \mathscr{E}_0) \cong \varinjlim_i H^0(X \times S, \mathscr{E}_i)$. We define as in lemma 5.12 a $R\{\tau\}$-module structure on $H^0(U \times S, \mathscr{E}_0)$. If $m \in H^0(X \times S, \mathscr{E}_i)$ we define

$$\tau m := t_i({}^{\tau}m) \in H^0(X \times S, \mathscr{E}_{i+1}).$$

If $m_0 \in H^0(X \times S, \mathscr{E}_0)$ is a base element then the map

$$\begin{aligned} R\{\tau\} &\longrightarrow H^0(U \times S, \mathscr{E}_0) \\ 1 &\longmapsto m_0 \end{aligned}$$

defines by corollary 5.19 an isomorphisms of $R\{\tau\}$-modules. □

5.3.2 Construction of a Drinfeld module

Proposition 5.21
Let $(\mathscr{E}_i, s_i, t_i)$ be an object of the category $\mathscr{E}\ell\ell_X^{(d)}(S)$ and let the conditions of lemma 5.19 be satisfied. Then $(\mathscr{E}_i, s_i, t_i)$ defines a standard Drinfeld module of rank d.

Proof We use theorem 4.42 an get projective R-modules (M, M_i) for all $i \in \mathbb{Z}$ such that the conditions of category II' are satisfied. By corollary 5.19 for all $i \in \mathbb{Z}$ the R-modules M_i are free of rank $\max(0, i+1)$. By corollary 5.20 the choice of a base element $m_0 \in M_0$ defines an isomorphism $M \cong R\{\tau\}$ of $R\{\tau\}$-modules. Using

$$\begin{array}{ccccc} A & \longrightarrow & M & \xrightarrow{\cong} & R\{\tau\} \\ a & \longmapsto & am_0 & \longrightarrow & \varphi(a) \end{array}$$

we define a $\mathbb{F}_q$-linear map $\varphi : A \longrightarrow R\{\tau\}$ such that

$$\deg_\tau \varphi(a) \leq d \deg(\infty) \deg(a)$$

for all $0 \neq a \in A$. As by assumption for all $i \in \mathbb{Z}$ the maps t_i are $\mathcal{O}_X$-linear we conclude

$$a(\tau m) = \tau(am)$$

for all $m \in M$. We get $\varphi(ab) = \varphi(a)\varphi(b)$ that is φ is a ringhomomorphism. Further on we get by the conditions of category II' for each element $0 \neq a \in A$ an isomorphism

$$M_0 \xrightarrow{a} M_{d \deg \infty \deg(a)} / M_{d \deg \infty \deg(a)-1}.$$

Let be

$$R\{\tau\}_{\leq n} := \{f(\tau) \in R\{\tau\} \mid \deg_\tau f(\tau) \leq n\}$$

for all $n \in \mathbb{N}$. The isomorphism $M \cong R\{\tau\}$ induces by construction a R-linear isomorphism

$$\begin{aligned} M_{d \deg \infty \deg a} / M_{d \deg \infty \deg(a)-1} &\cong R\{\tau\}_{\leq d \deg(\infty) \deg a} / R\{\tau\}_{\leq d \deg(\infty) \deg a - 1} \\ &\cong R. \end{aligned}$$

We define $\varphi(a) := \sum_{n=0}^{d \deg(\infty) \deg(a)} r_n \tau^n$. and get a R-linear isomorphism

$$\begin{aligned} M_0 &\xrightarrow{a} R \\ m_0 &\longmapsto r_{d \deg \infty \deg a}. \end{aligned}$$

As m_0 is generating the R-module M_0 we conclude that $r_{d \deg \infty \deg a}$ is a unit in R. In particular the above constructed Drinfeld module is standard of of rank d. The characteristic of the Drinfeld module is given by the composition of the map φ and the map $\partial : R\{\tau\} \longrightarrow R$. □

5.3.3 Drinfeld modules $\Longrightarrow$ Elliptic sheaves

The conditions 1), 2) and 3) of an elliptic sheaf (cf. 5.1) are given by the construction of section 3 and remark 3.19. Further on we get from the construction that the diagram in a) commutes.

for b), c) It is $(\mathrm{pr}_X^* \mathcal{O}_X(\infty))|_{\mathrm{spec}(A) \times S} \cong \mathcal{O}_{\mathrm{spec}\, A \times S}$ and the assertion follows form the construction and corollary 3.17.

for d) By construction we have $\mathrm{supp}\, \mathscr{E}_i / \mathscr{E}_{i-1} \subseteq \mathrm{spec}\, \mathcal{O}_{X,\infty} \times S$. The assertion follows now from 2) in the proof of proposition 3.10 and lemma 5.3.

for e) Corollary 3.16 implies that it is $\operatorname{supp} \mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} \subseteq \operatorname{spec} A \times S$. The assertion follows now from the isomorphisms

$$\operatorname{pr}_{S*} \mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} \cong \operatorname{pr}_{S*} (\mathscr{E}_i/{}^\tau\mathscr{E}_{i-1}|_{\operatorname{spec} A\times S}) \cong \operatorname{pr}_{S*} \mathscr{E}_0 \cong R$$

for all $i \in \mathbb{Z}$.

5.4 Arbitrary base scheme

We will construct below the correspondence between Drinfeld modules and elliptic sheaves over an arbitrary base scheme S. For this we need the following lemma.

Lemma 5.22
It is

$$\operatorname{pr}_{S*} (\mathscr{E}_0|_{\operatorname{spec} A\times S}) \cong \bigoplus_{n=0}^{\infty} (\operatorname{pr}_{S*} \mathscr{E}_0)^{\tau^n}.$$

Proof By proposition 2.67 it is

$$\operatorname{pr}_{S*} (\mathscr{E}_0|_{\operatorname{spec} A\times S}) \cong \varinjlim_{i\geq 0} \operatorname{pr}_{S*} \mathscr{E}_i.$$

By assumption for all $i \in \mathbb{Z}$ exists an exact sequence

$$0 \longrightarrow {}^\tau\mathscr{E}_{i-1} \longrightarrow \mathscr{E}_i \longrightarrow \mathscr{E}_i/{}^\tau\mathscr{E}_{i-1} \longrightarrow 0.$$

If we apply the functor pr_{S*} then the sequence stays exact for $i \geq 0$. Using the embedding

$$s_{i-1} \circ \cdots \circ s_0 : \mathscr{E}_0 \longrightarrow \mathscr{E}_i$$

and corollary 5.16 we get the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{pr}_{S*} {}^\tau\mathscr{E}_{i-1} & \longrightarrow & \operatorname{pr}_{S*} \mathscr{E}_i & \longrightarrow & \operatorname{pr}_{S*} (\mathscr{E}_i/{}^\tau\mathscr{E}_{i-1}) & \longrightarrow & 0 \\ & & & & & \nwarrow & \| & & \\ & & & & & & \operatorname{pr}_{S*} \mathscr{E}_0. & & \end{array}$$

In particular the sequence splits and we get for all $i \geq 0$

$$\operatorname{pr}_{S*} \mathscr{E}_i \cong \operatorname{pr}_{S*} {}^\tau\mathscr{E}_{i-1} \oplus \operatorname{pr}_{S*} \mathscr{E}_0.$$

By corollary 2.50 it is $\operatorname{pr}_{S*} {}^\tau\mathscr{E}_{i-1} \cong {}^\tau(\operatorname{pr}_{S*} \mathscr{E}_{i-1})$. Going to the inductive limit we get the assumption. □

5.4.1 Drinfeld modules $\Longrightarrow$ Elliptic sheaves

Let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be a standard Drinfeld module over S. As in section 3.8 we construct for all $i \in \mathbb{Z}$ the $\mathcal{O}_{X\times S}$-vector bundles

$$\mathscr{E}_i := \operatorname{Proj}\left(\bigoplus_{k=0}^{\infty} \bigoplus_{n=0}^{i+kd\deg\infty} \mathscr{L}^{q^{-n}}\right).$$

For all $i \in \mathbb{Z}$ the canonical embeddings

$$\bigoplus_{n=0}^{i+kd\deg\infty} \mathscr{L}^{-q^n} \hookrightarrow \bigoplus_{n=0}^{i+1+kd\deg\infty} \mathscr{L}^{-q^n}$$

induce homogeneous $\mathcal{S}_X \otimes \mathcal{O}_S$-linear maps

$$\mathcal{M}_i \hookrightarrow \mathcal{M}_{i+1}$$

of degree 0 that is $\mathcal{O}_{X\times S}$-linear maps

$$s_i : \mathscr{E}_i \hookrightarrow \mathscr{E}_{i+1}.$$

It is for all $i \in \mathbb{Z}$

$$\left(\bigoplus_{n=0}^{i+kd\deg\infty} \mathscr{L}^{-q^n}\right)^{\tau} \cong \bigoplus_{n=0}^{i+kd\deg\infty} \mathscr{L}^{-q^{n+1}} \cong \bigoplus_{n=1}^{i+1+kd\deg\infty} \mathscr{L}^{-q^n}.$$

From the canonical inclusions

$$\bigoplus_{n=1}^{i+1+kd\deg\infty} \mathscr{L}^{-q^n} \hookrightarrow \bigoplus_{n=0}^{i+1+kd\deg\infty} \mathscr{L}^{-q^n}$$

we get homogeneous $\mathcal{S}_X \otimes \mathcal{O}_S$-linear maps

$$\mathcal{M}_i^{\tau} \hookrightarrow \mathcal{M}_{i+1}$$

of degree 0. They induce $\mathcal{O}_{X\times S}$-linear maps

$$t_i : {}^{\tau}\mathscr{E}_i \hookrightarrow \mathscr{E}_{i+1}.$$

For all $j \in \mathbb{Z}$ the canonical maps

$$\mathcal{S}_X[1]_j \otimes_{\mathcal{S}_X} \bigoplus_{n=0}^{i+kd\deg\infty} \mathscr{L}^{-q^n} \longrightarrow \bigoplus_{n=0}^{i+kd\deg\infty+(j+1)d\deg\infty} \mathscr{L}^{-q^n}$$

induce homogeneous $\mathcal{S}_X \otimes \mathcal{O}_S$-linear maps

$$\mathcal{S}_X[1] \otimes_{\mathcal{S}_X} \mathcal{M}_i \longrightarrow \mathcal{M}_{i+d \deg \infty}$$

of degree 0. They induce $\mathcal{O}_{X\times S}$-linear maps

$$\mathcal{O}_{X\times S}(\infty) \otimes_{\mathcal{O}_{X\times S}} \mathscr{E}_i \longrightarrow \mathscr{E}_{i+d \deg \infty}.$$

The construction above is locally in S equal to the construction in section 5.3.3. The conditions of the category $\mathscr{E}\ell\ell_X^{(d)}(S)$ are local in S. In this case the conditions are already checked in section 5.3.3.

5.4.2 Elliptic sheaves $\Longrightarrow$ Drinfeld modules

Let $(\mathscr{E}_i, s_i, t_i)$ be an elliptic sheaf of the category $\mathscr{E}\ell\ell_X^{(d)}(S)$. Define $U := \operatorname{spec} A \subseteq X$. By lemma 5.22 we have

$$\operatorname{pr}_{S*}(\mathscr{E}_0|_{U\times S}) \cong \bigoplus_{i\geq 0} (\operatorname{pr}_{S*}\mathscr{E}_0)^{\tau^n}.$$

Let

$$\mathscr{L}^{-1} := \operatorname{pr}_{S*}\mathscr{E}_0.$$

Then we have

$$\operatorname{pr}_{S*}\mathscr{E}_0|_{U\times S} \cong \bigoplus_{n\geq 0} \mathscr{L}^{-q^n} \cong \mathscr{H}om_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/\mathcal{O}_S}).$$

Let $a \in A$. Then a induces an $\mathcal{O}_S$-linear map

$$\operatorname{pr}_{S*}(\mathscr{E}_0|_{U\times S}) \xrightarrow{\ a\ } \operatorname{pr}_{S*}(\mathscr{E}_0|_{U\times S}).$$

We get the diagram

$$\begin{array}{ccccc} & & \operatorname{pr}_{S*}(\mathscr{E}_0|_{U\times S}) & \xrightarrow{a} & \operatorname{pr}_{S*}(\mathscr{E}_0|_{U\times S}) \\ & & \Vert & & \Vert \\ \mathscr{L}^{-1} & \longrightarrow & \bigoplus_{n\geq 0} \mathscr{L}^{-q^n} & \longrightarrow & \bigoplus_{n\geq 0} \mathscr{L}^{-q^n} \end{array}$$

and from this a map

$$A \longrightarrow \mathscr{E}nd_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}).$$

By composing the canonical projection $\bigoplus_{n\geq 0} \mathscr{L}^{-q^n} \longrightarrow \mathscr{L}^{-1}$ we get a map

$$\operatorname{char} : A \longrightarrow \operatorname{End}_{\mathcal{O}_S}(\mathscr{L}^{-1}) \cong \mathcal{O}_S(S).$$

This is the characteristic.
The construction above is locally in S equal to the construction in section 5.3.1. The conditions of a Drinfeld module are local in S. In this case the conditions are already checked in section 5.3.1.

5.4.3 Drinfeld modules $\Longrightarrow$ Category II'

In the construction of an elliptic sheaf of section 5.4.1 we implicit construct an object of category II'. We describe this construction in detail below.
Let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be a standard Drinfeld module. Let $\mathscr{L}$ be the associated $\mathcal{O}_S$-line bundle of E. We define

$$\mathscr{M}_i := \bigoplus_{n=0}^{i} \mathscr{L}^{-q^n}$$

for all $i \geq 0$. For $i < 0$ let $\mathscr{M}_i = 0$. For all $a \in A$ the map

$$e_a : \mathscr{L}^{-1} \longrightarrow \bigoplus_{n=0}^{d \deg \infty \deg a} \mathscr{L}^{q^{-n}}$$

defines $\mathcal{O}_S$-linear morphisms

$$\mathscr{M}_i \xrightarrow{\ a\ } \mathscr{M}_{i+d \deg \infty \deg a}.$$

By definition of a standard Drinfeld module for all $0 \neq a \in A$ the composition of the maps

$$\mathscr{L}^{-1} \xrightarrow{e_a} \bigoplus_{n=0}^{d \deg \infty \deg a} \mathscr{L}^{q^{-n}} \xrightarrow{\text{pr}} \mathscr{L}^{q^{-d \deg \infty \deg a}}$$

is an isomorphism. We conclude that for all $i \geq 0$ the map

$$\mathscr{M}_i / \mathscr{M}_{i-1} \xrightarrow{\ a\ } \mathscr{M}_{i+d \deg \infty \deg a} / \mathscr{M}_{i-1+d \deg \infty \deg a}$$

is an isomorphism too. From this we get an object of category II'.
Let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ and E' $= (\mathbb{G}_{a/\mathscr{L}'}, e')$ be standard Drinfeld modules of rank d. Let $\varphi \in \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a/\mathscr{L}}, \mathbb{G}_{a/\mathscr{L}'})$ be a morphism of Drinfeld modules. By proposition 3.1 we have

$$\varphi = \oplus_{m=0}^{\infty} \varphi_m \qquad \text{s.t.} \qquad \varphi_m : \mathscr{L}'^{-1} \longrightarrow \mathscr{L}^{-q^m}.$$

For all $a \in A$ and all $m \in \mathbb{N}$ we get the commutative diagram

$$\begin{array}{ccc}
\mathscr{L}^{-q^m} & \xrightarrow{\ e_a\ } & \bigoplus_{n=0}^{d \deg \infty \deg a} \mathscr{L}^{-q^{n+m}} \\
\Big\uparrow \varphi_m & & \Big\uparrow \varphi_m \\
\mathscr{L}'^{-1} & \xrightarrow{\ e'_a\ } & \bigoplus_{n=0}^{d \deg \infty \deg a} \mathscr{L}'^{-q^n}.
\end{array}$$

Let $(\mathscr{M}_i, \mathscr{M})$ and $(\mathscr{M}'_i, \mathscr{M}')$ be objects of category II' associated to the Drinfeld modules E and E'. Then we get a morphisms

$$(\mathscr{M}'_i, \mathscr{M}') \xrightarrow{\varphi} (\mathscr{M}_i, \mathscr{M})$$

in the category II'.
In addition we get a contravariant faithfull functor

$$\left\{ \begin{array}{l} \text{Standard Drinfeld modules} \\ \text{over } S \text{ of rank } d \end{array} \right\} \leftrightarrow \{ \text{ Category II' } \}.$$

5.4.4 Equivalences of the categories

Finally we will complete the proof of theorem 5.10.
By the constructions in section 5.4.1 and section 5.4.2 we get functors between the category of standard Drinfeld modules of rank d and the category $\mathscr{E}\ell\ell_X^{(d)}(S)$.
The comparison of the category of Drinfeld modules with category II' (section 5.4.3) and between category I' with category II' (theorem 4.42) shows that the both categories are functorial equivalent.

6 Division points and level structures

6.1 Division points and level structures of Drinfeld modules

Let S be a scheme over the finite field $\mathbb{F}_q$ and let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be a Drinfeld module over S. If $S = \operatorname{spec} R$ is an affine scheme then we call by $e_a \in R\{\tau\}$ the image of a under the map e for all $a \in A$. We can regard each element $f \in R\{\tau\}$ as an additive polynomial. We notate f in this case $f(X) \in R[X]$. Let $0 \neq I \subsetneq A$ be an ideal.

Definition 6.1
Let $E[I]$ be the contravariate functor on the category of schemes over S with image in the category of A/I modules defined by

$$T/S \longmapsto \{x \in E(T) \mid Ix = 0\} = Hom_A(A/I, E(T))$$

for all schemes T/S.

Eigenschaften 6.2

1) $E[I] \subseteq E$ is a closed (sub-)group scheme. If $I = (a_1, \dots a_n)$ for appropriate elements $a_1, \dots, a_n \in A$ then it is

$$E[I] = \operatorname{Ker}(E \xrightarrow{e_{a_1}, \dots, e_{a_n}} E \times_S \cdots \times_S E).$$

In the affine case $S = \operatorname{spec} R$ we have

$$E[I] = \operatorname{spec} R[X]/(e_{a_1}(X), \dots, e_{a_n}(X)).$$

2) If I, J are coprime ideals in A then it is

$$E[IJ] \cong E[I] \times_S E[J].$$

3) The group scheme $E[I]$ is finite and flat over S of rank $|A/I|^d$.

4) If I is coprime to characteristic of the Drinfeld module E then $E[I]$ is étale over S.

5) The group scheme $E[I]$ is compatible with base change that is for each scheme T/S we have

$$E[I] \times_S T \cong (E \times_S T)[I].$$

Proof Cf. [Leh00], chapter 2, proposition 4.1, page 27 et seq. □

If $S = \operatorname{spec} L$ is a field then the characteristic of the Drinfeld module (L, e) is a prime ideal of A. Thus we can defined the *height* of (L, e). It is called h. In the case of an algebraic closed field we have the following explicit description of the I division points:

Proposition 6.3
Let $\mathfrak{p} \subseteq A$ be a prime ideal and let $I = \mathfrak{p}^n$ for a $n > 0$. Then we have

$$E[\mathfrak{p}^n](L) \cong \begin{cases} (\mathfrak{p}^{-n}/A)^d & \text{for } \mathfrak{p} \neq \operatorname{char} E \\ (\mathfrak{p}^{-n}/A)^{d-h} & \text{for } \mathfrak{p} = \operatorname{char} E. \end{cases}$$

Proof Cf. [Leh00], chapter 2, corollary 2.4, page 24. □

Definition 6.4 ([Dri76])
Let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be a Drinfeld module of rank d over S and let $0 \neq I \subsetneq A$ be an ideal. A Level I structure is an A linear map

$$\iota : (I^{-1}/A)^d \longrightarrow E(S),$$

such that for all prime ideals $0 \neq \mathfrak{p} \in V(I)$ there exists an identity of the Cartier divisors

$$E[\mathfrak{p}] = \sum_{\alpha \in (\mathfrak{p}^{-1}/A)^d} \iota(\alpha).$$

Remark 6.5
1) If I is coprime to $\operatorname{char} E$ then a level I structure is an isomorphism of group schemes

$$(I^{-1}/A)^d_S \simeq E[I].$$

2) If ι is a level I structure then we have the identity of Cartier divisors

$$E[I] = \sum_{\alpha \in (I^{-1}/A)^d} \iota(\alpha).$$

Proof Cf. [Leh00], chapter 2, example 4.3, page 29 and chapter 3, proposition 3.3, page 49. □

Lemma 6.6
Let R be an $\mathbb{F}_q$ algebra. Let $H \subset \mathbb{G}_{a/R}$ be a finite flat subgroup scheme of rank n over R. Then there is a uniquely defined normalized additive polynomial $h \in R[X]$ of degree n such that $H = V(h)$.

Proof Cf. [Leh00], chapter 1, lemma 3.3, page 9. □

Let (R, e) be a Drinfeld module of rank d over R and let $I \subsetneq A$ be an ideal. By lemma 6.6 there exists an additive polynomial $h_I \in R[X]$ of rank $|I^{-1}/A|^d$ such that $E[I] = V(h_I)$. The equality of Cartier divisors in the definition of level I structures denotes in this case the equality of polynomials

$$h_{\mathfrak{p}} = \prod_{\alpha \in (\mathfrak{p}^{-1}/A)^d} (X - \iota(\alpha))$$

for all $\mathfrak{p} \supset I$.

Proposition 6.7
Let R be a reduced ring. Let

$$\iota : (I^{-1}/A)^d \longrightarrow E(R)$$

be an A-linear map. If it is $h_I = \prod_{\alpha \in (I^{-1}/A)^d}(X - \iota(\alpha))$ then ι is a level I structure.

Remark 6.8
If char E is coprime to I then the assertion follows from remark 6.5, 1).

Proof (of proposition 6.7) We first show the case $R = L$ is a field. From 6.2, 2) we conclude that it is sufficient to concider the case $I = \mathfrak{p}^n$.
By assumption we have $h_{\mathfrak{p}^n} = \prod_{\alpha \in (\mathfrak{p}^{-n}/A)^d}(X - \iota(\alpha))$. By proposition 6.3 it is

$$E[\mathfrak{p}^n](L) \cong (\mathfrak{p}^{-n}/A)^{d-h}.$$

We get the exact sequence

$$0 \longrightarrow \ker \iota \longrightarrow (\mathfrak{p}^{-n}/A)^d \xrightarrow{\iota} (\mathfrak{p}^{-n}/A)^{d-h} \longrightarrow 0.$$

of $A/\mathfrak{p}^n$ modules. In particular $\ker \iota \cong (\mathfrak{p}^{-n}/A)^h$ is a free $A/\mathfrak{p}^n$ module. For all $n \in \mathbb{N}$ define $k_{\mathfrak{p}^n} := |\mathfrak{p}^{-n}/A|^h$. Then it is

$$h_{\mathfrak{p}^n} = \prod_{l \in \operatorname{Im} \iota} (X - l)^{k_{\mathfrak{p}^n}}.$$

By the canonical embedding $\mathfrak{p}^{-1}/A \hookrightarrow \mathfrak{p}^{-n}/A$ we define the polynomial

$$\tilde{h}_{\mathfrak{p}} := \prod_{\alpha \in (\mathfrak{p}^{-1}/A)^d} (X - \iota|_{(\mathfrak{p}^{-1}/A)^d}(\alpha)).$$

Now we have to show that it is $\tilde{h}_{\mathfrak{p}} = h_{\mathfrak{p}}$. As we have $\operatorname{Ker} \iota \cong (\mathfrak{p}^{-n}/A)^h$ we conclude

$$\operatorname{Ker} \iota|_{(\mathfrak{p}^{-1}/A)^d} \cong (\mathfrak{p}^{-1}/A)^h.$$

Furthermore

$$\tilde{h}_{\mathfrak{p}} = \prod_{l \in \operatorname{Im} \iota|_{(\mathfrak{p}^{-1}/A)^d}} (X - l)^{k_{\mathfrak{p}}}.$$

By definition we have $E[\mathfrak{p}] = V(h_{\mathfrak{p}})$. In particular by the A linearity of the map ι each $l \in \operatorname{Im} \iota|_{(\mathfrak{p}^{-1}/A)^d}$ is a zero of the polynomial $h_{\mathfrak{p}}$ By proposition 2.3 we have $E[\mathfrak{p}](L) \cong (\mathfrak{p}^{-1}/A)^h$. The additive polynomial $h_{\mathfrak{p}}$ possesses exactly $|\mathfrak{p}^{-1}/A|^{d-h}$ different zeros and each zero has multiplicity $k_{\mathfrak{p}}$. This proofs the assumption.
If R is a domain then we conclude the assumption by using the quotient field of R. If R is a reduced ring Ring then we can test the assumption modulo all prime ideals $\mathfrak{P}$ of R. Then we first get the assumption for the domains $R/\mathfrak{P}$. In the case of a reduced ring we have in particular $\bigcap_{\mathfrak{P} \in \operatorname{spec} R} = (0)$ and we get again the assumption. □

Remark 6.9
Im Falle eines beliebigen Ringes R ist es dem Autor trotz einiger Bemühungen leider nicht gelungen, einen (elementaren) Beweis von Satz 6.7 zu finden. Auch ein Gegenbeispiel ließ sich nicht konstruieren. Manche Autoren (z.B. [DH87], Definition 6.1) verwenden Satz 6.7 als Definition von Level-I-Strukturen. Eine Klärung des Problems wäre daher wünschenswert.
Vielleicht würde eine genaue Analyse des Beweises von [Leh00], Kapitel 3, Satz 3.3, einen Beweis des Satzes ermöglichen. Insbesondere müssten die benötigten Deformationsargumente auf den obigen Fall übertragen werden.

6.2 F-sheaves

Let S be a scheme over $\mathbb{F}_q$ and let $\mathscr{E}$ be $\mathcal{O}_S$ vector bundle of rank n. We denote by $\mathscr{E}^\vee := \mathscr{H}om_{\mathcal{O}_S}(\mathscr{E}, \mathcal{O}_S)$ the dual vector bundle.

Definition 6.10 ([Dri87])
We call an $\mathcal{O}_S$ linear map

$$\varphi : \mathscr{E}^\tau \longrightarrow \mathscr{E}$$

a Frobenius structure *on $\mathscr{E}$ and we denote the pair $(\mathscr{E}, \varphi)$ a F sheaf. The vectorbundles equipped with a Frobenius structure build in an obvious way a category. This is is category of F-sheaves over S.*

Remark 6.11
The Frobenius homomorphism $\mathscr{E}^\vee \longrightarrow {}^\tau\mathscr{E}^\vee$ induces a group homomorphism

$$\operatorname{Frob}^g : \mathbb{G}_{a/\mathscr{E}^\vee} \longrightarrow \mathbb{G}_{a/{}^\tau\mathscr{E}^\vee}.$$

Definition 6.12 ([Dri87])
Let $(\mathscr{E},\varphi)$ be a Frobenius structure. It induces a group homomorphism

$$\varphi^g : \mathbb{G}_{a/\mathscr{E}^\vee} \longrightarrow \mathbb{G}_{a/{}^\tau\mathscr{E}^\vee}.$$

Let be

$$\mathrm{Gr}(\mathscr{E}) := \mathrm{Ker}(\mathbb{G}_{a/\mathscr{E}^\vee} \xrightarrow{\varphi^g-\mathrm{Frob}^g} \mathbb{G}_{a/{}^\tau\mathscr{E}^\vee}).$$

Remark 6.13
Let $(\mathscr{E},\varphi)$ be a F-sheaf of rank n.

1) By consruction Gr is a functor from the category of F-sheaves into the category of commutative group schemes over S.

2) The scheme $\mathrm{Gr}(\mathscr{E})$ is a finite flat group scheme over S of rank q^n.

3) The scheme $\mathrm{Gr}(\mathscr{E})$ is étale over S iff φ is an isomorphism.

4) The functor Gr is exact and fully faithfull.

Proof Cf. [Dri87], proposition 2.1, page 110. □

Example 6.14
Let be $\mathscr{E} \cong \mathcal{O}_S^{\oplus n}$. The canonical isomorphism $\varphi : {}^\tau\mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S^{\oplus n}$ defines a Frobenius structure on $\mathcal{O}_S^{\oplus n}$. For all schemes T/S we get

$$\begin{aligned}\mathrm{Gr}(\mathcal{O}_S^{\oplus n})(T) &= \{\alpha \in \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T^n, \mathcal{O}_T) \mid \alpha^q = \alpha\} \\ &= (\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, \mathbb{F}_q))_S(T) = (\mathbb{F}_q^{n\vee})_S(T).\end{aligned}$$

We conclude $\mathrm{Gr}(\mathcal{O}_S^{\oplus n}) \cong (\mathbb{F}_q^{n\vee})_S$.

Example 6.15
Let B be a finite $\mathbb{F}_q$ algebra of rank n. Let be $\mathscr{E} := \mathcal{O}_S \otimes B$. As in the previous example we get by the isomorphism ${}^\tau\mathcal{O}_S \cong \mathcal{O}_S$ a Frobenius structure on $\mathscr{E}$. Let T/S be a scheme. We conclude

$$\begin{aligned}\mathrm{Gr}(\mathscr{E})(T) &= \{\alpha \in \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T \otimes B, \mathcal{O}_T) \mid \alpha^q = \alpha\} \\ &= \mathrm{Hom}_{\mathbb{F}_q}(B, \mathcal{O}_T(T)) \\ &= (B^\vee)_S(T).\end{aligned}$$

The group schemes $\mathrm{Gr}(\mathscr{E})$ and $(B^\vee)_S$ are in a natural way B modules and the isomorphism $\mathrm{Gr}(\mathscr{E}) \cong (B^\vee)_S$ is B linear.

6.3 Division points and level structures of elliptic sheaves

Let $(\mathscr{E}_i, s_i, t_i)$ be an elliptic sheave in the category $\mathscr{E}\ell\ell_X^{(d)}(S)$. Let $0 \neq I \subsetneq A$ be an ideal and let $V(I) \subseteq U = \operatorname{spec} A$ be the associated closed subscheme of U.
We define

$$\mathscr{F} := \operatorname{pr}_{S*} \mathscr{E}_i|_{U \times S}, \quad \mathscr{F}_I := \operatorname{pr}_{S*} \mathscr{E}_i|_{V(I) \times S}.$$

The definition are independent of the choice of the number $i$as by assumption it is $\mathscr{E}_i|_{U\times S} \cong \mathscr{E}_{i+1}|_{U\times S}$ for all $i \in \mathbb{Z}$. We conclude that $\mathscr{F}_I$ is an $\mathcal{O}_S$ vector bundle of rank $n := |A/I|^d$. In particular the $\mathcal{O}_X$ module structure on $\mathscr{E}$ induces an A module structure on $\mathscr{F}_I$. Let $\mathcal{O}_S\{\tau\}$ be the sheaf of $\mathcal{O}_S$ algebras defined by

$$\mathcal{O}_S\{\tau\}(V) := \mathcal{O}_S(V)\{\tau\}$$

for all open subsets $V \subseteq S$. By the $\mathcal{O}_S$ linear map $t : {}^\tau\mathscr{F}_I \longrightarrow \mathscr{F}_I$ becomes $\mathscr{F}_I$ a φ-sheaf.
It is

$$\mathscr{F}_I \cong \bigoplus_{n=0}^{\infty} \mathscr{L}^{-q^n} \otimes_A A/I.$$

Let

$$\bigoplus_{n=0}^{\infty} \mathscr{L}^{-q^n} \longrightarrow \bigoplus_{n=0}^{\infty} \mathscr{L}^{-q^n} \otimes_A A/I$$

be the canonical projection map.
We conclude

$$\operatorname{Gr}(\mathscr{F}_I)(T) = \operatorname{Hom}_{\mathcal{O}_T\{\tau\}}(\mathscr{F}_T, \mathcal{O}_T) \hookrightarrow \operatorname{Hom}_{\mathcal{O}_T\{\tau\}}(\bigoplus_{n=0}^{\infty} \mathscr{L}_T^{-q^n}, \mathcal{O}_T) \cong \mathscr{L}(T).$$

We get a canonical A linear group homomorphism

$$\operatorname{Gr}(\mathscr{F}_I) \hookrightarrow \mathbb{G}_{a/\mathscr{L}}.$$

Definition 6.16
A level I structure on an elliptic sheaf $(\mathscr{E}_i, s_i, t_i)$ is an A linear map

$$\iota : (A/I)^{\vee d} \longrightarrow \mathscr{L}(S),$$

such that for all prime ideals $\mathfrak{p} \supseteq I$ there is an equality of the divisors

$$\prod_{\alpha \in (A/\mathfrak{p})^{\vee d}} (X - \iota(\alpha)) = \operatorname{Gr}(\mathscr{F}_\mathfrak{p}).$$

Remark 6.17
If the characteristic of the elliptic sheaf and $V(I)$ are disjoint then $\mathrm{Gr}\,\mathscr{F}_I$ is an étales group scheme. In this case definition 6.16 is equivalent to the declaration of an isomorphism

$$\iota : ((A/I)^{\vee d})_S \longrightarrow \mathrm{Gr}\,\mathscr{F}_I$$

(cf. remark 6.5, 1)).

Remark 6.18
In the case that the characteristic of the elliptic sheaf and $V(I)$ are disjoint the article [BS97] defines a level I structure as follows. A level I structure is an isomorphism ι such that the diagram

$$\begin{array}{ccc} {}^{\tau}\mathscr{F}_I & \xrightarrow{\quad t|_I \quad} & \mathscr{F}_I \\ {\scriptstyle {}^{\tau}\iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ (A/I)^d \otimes {}^{\tau}\mathcal{O}_S & \xrightarrow{\cong} & (A/I)^d \otimes \mathcal{O}_S \end{array}$$

commutes. So the map ι is an isomorphism of the F-sheaves $(\mathscr{F}_I, t|_I)$ and $((A/I)^d \otimes \mathcal{O}_S, \cong)$ (cf. example 6.14). We show below the equivalence of the two definitions:

1) Using the functor Gr gives an isomorphism

$$((A/I)^{\vee d})_S \longrightarrow \mathrm{Gr}\,\mathscr{F}_I$$

(cf. example 6.14).

2) By remark 6.13 the functor Gr is fully faithfull. For this the isomorphism $((A/I)^{\vee d})_S \longrightarrow \mathrm{Gr}\,\mathscr{F}_I$ induces an isomorphism of the corresponding F-sheaves.

6.4 Comparison of both concepts

We show below that a level I structure of a Drinfeld module and a level I structure of an elliptic sheaf are equivalent concepts. First we need the following two lemmas:

Lemma 6.19
Let $0 \neq I \subsetneq A$ be an ideal. Then the A/I modules A/I and $\mathrm{Hom}_{\mathbb{F}_q}(A/I, \mathbb{F}_q)$ are isomorphic.

Proof It is sufficient to proof the assumption for the ideals $I = \mathfrak{p}^e$. For this let be $e \in \mathbb{N}$, let $0 \neq \mathfrak{p}$ be a prime ideal of A and let $\pi_\mathfrak{p} \in A$ be a uniformizing element of the prime ideal $\mathfrak{p}$. Then the elements

$$\overline{1}, \overline{\pi}_\mathfrak{p}, \ldots, \overline{\pi}_\mathfrak{p}^{e-1} \in A/\mathfrak{p}^e$$

form a $\mathbb{F}_q$ base of $A/\mathfrak{p}^e$. Let $\psi \in \mathrm{Hom}_{\mathbb{F}_q}(A/\mathfrak{p}^e, \mathbb{F}_q)$ be such that

$$\psi(\overline{1}) = 1, \ldots, \psi(\overline{\pi}_\mathfrak{p}^{e-1}) = 1.$$

We define the map

$$\begin{aligned} \alpha : A/\mathfrak{p}^e &\longrightarrow \mathrm{Hom}_{\mathbb{F}_q}(A/\mathfrak{p}^e, \mathbb{F}_q) \\ \overline{a} &\longmapsto \overline{a}\psi. \end{aligned}$$

An easy calculation shows, that the map is $A/\mathfrak{p}^e$-linear, well defined an injective. As an injective map between finite sets is allways surjectiv we get the assumption. □

Lemma 6.20
Let $0 \neq I \subsetneq A$ be an ideal. Then the A/I-modules A/I and I^{-1}/A are isomorphic.

Proof Let $I = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$ be the prime decomposition of the ideal I. There exists an element $x \in A$ such that $v_{\mathfrak{p}_1}(x) = e_1, \ldots, v_{\mathfrak{p}_s}(x) = e_s$. By

$$\begin{aligned} I^{-1}/A &\longrightarrow A/I \\ a &\longmapsto xa \end{aligned}$$

we get an A/I-linear map. The map is an isomorphism, as by the choice of x the map is locally an isomorphism at primes of A/I. □

Proposition 6.21
Let $(\mathscr{E}_i, s_i, t_i)$ be an elliptic sheaf in the category $\mathscr{E}\ell\ell_X^{(d)}(S)$ and let $E = (\mathbb{G}_{a/\mathscr{L}}, e)$ be the corresponding Drinfeld module over S of rank d. Let $0 \neq I \subsetneq A$ be an ideal and let $\mathscr{F}_I$ be the corresponding F-sheaf. Then a level I structure of an elliptic sheaf $(\mathscr{E}_i, s_i, t_i)$ and a level I structure of a Drinfeld module E are functorial equivalent.

Proof By section 6.3 we have for all schemes T/S

$$\mathrm{Gr}(\mathscr{F}_I)(T) = \mathrm{Hom}_{\mathcal{O}_T\{\tau\}}(\mathscr{F}_T, \mathcal{O}_T) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_T\{\tau\}}(\bigoplus_{n=0}^{\infty} \mathscr{L}_T^{-q^n}, \mathcal{O}_T) \cong \mathscr{L}(T).$$

By

$$\mathrm{Hom}_{\mathcal{O}_T\{\tau\}}(\mathscr{F}_T, \mathcal{O}_T) = \left\{ x \in \mathrm{Hom}_{\mathcal{O}_T\{\tau\}}(\bigoplus_{n=0}^{\infty} \mathscr{L}_T^{-q^n}, \mathcal{O}_T) \mid Ix = 0 \right\}$$

we get the description

$$\mathrm{Gr}(\mathscr{F}_I)(T) = \{x \in \mathscr{L}(T) \mid Ix = 0\}.$$

By the definition of the I-division points $E[I]$ of a Drinfeld module we have

$$E[I](T) = \{x \in \mathscr{L}(T) \mid Ix = 0\}$$

and there is a canonical isomorphism of group schemes

$$\mathrm{Gr}(\mathscr{F}_I) \cong E[I].$$

By lemma 6.19 and lemma 6.20 there exists an A/I-linear isomorphism

$$I^{-1}/A \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_q}(A/I, \mathbb{F}_q).$$

Thus we get the assumption. □

By proposition 6.21 we get the main result:

Theorem 6.22
Let $0 \neq I \subsetneq A$ be an ideal. There is an equivalence of categories:

$$\left\{ \begin{array}{l} \textit{Standard Drinfeld modules} \\ \textit{over } S \textit{ of rank } d \textit{ and a level} \\ I \textit{ structure} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \textit{Elliptic modules of the cate-} \\ \textit{gory } \mathscr{E}\ell\ell_X^{(d)}(S) \textit{ equipped with} \\ \textit{a level } I \textit{ structure} \end{array} \right\}$$

Remark 6.23
The correspondence of the categories depends on the choice of the isomorphism

$$I^{-1}/A \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_q}(A/I, \mathbb{F}_q).$$

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