Consecutive primes

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Theorem (Prime Number Theorem in Arithmetic Progressions)

Let $\pi(x; q, a)$ be the number of primes $p \leqslant x$ with $p \equiv a \mod q$. Then

$$\pi(x; q, a) := \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \mathbf{1}_{prime}(n) \sim \frac{\pi(x)}{\phi(q)} \sim \frac{x}{\phi(q) \log x}.$$

Here $\phi(q)$ is Euler's totient function, which counts the number of $a \bmod q$ with $\gcd(a, q) = 1$.

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Here $\phi(q)$ is Euler's totient function, which counts the number of a mod q with $\gcd(a, q) = 1$.
No dependence on a—primes are equidistributed mod q!
Primes are likely to be a mod q—a positive proportion of primes are a mod q.

How many pairs of consecutive primes are congruent to $(a, b) \mod q$?

A harder question:

How many primes congruent to a mod q are followed by primes congruent to b mod q? That is, how often does (a, b) appear in the sequence of primes mod q?

For fixed classes $a_1, \ldots, a_r \mod q$, how often does $(a_1, \ldots, a_r)$ appear in the sequence of primes mod q?

Example: $q = 10$

Are there infinitely many primes ending in 7 such that the next prime ends in 1? The first few examples are $(7, 11)$, $(37, 41)$, $(67, 71)$, $(97, 101)$, $(127, 131)$, $(277, 281)$, $(307, 311)$, $(317, 331)$, ...

In most cases, this question is open.

The consecutive primes conjecture

Conjecture: Consecutive primes equidistribute

For any $r \geqslant 1$, for any modulus q and for any $a_1, \ldots, a_r$ with $\gcd(a_i, q) = 1$ for all i,

$$\begin{aligned} \pi(x; q, (a_1, \ldots, a_r)) &:= \#\{p_n \leqslant x : p_{n+i} \equiv a_i \mod q \quad \forall 1 \leqslant i \leqslant r\} \\ &\sim \frac{\pi(x)}{(\phi(q))^r}. \end{aligned}$$

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For twin primes (primes p such that $p + 2$ is also prime) we think the following should be true:

The twin primes conjecture

There are infinitely many twin primes. More precisely, for some constant C_2,

$$\pi_{\text{twin}}(x) := \sum_{n \leqslant x} \mathbf{1}_{\text{prime}}(n) \mathbf{1}_{\text{prime}}(n+2) \sim C_2 \frac{\pi(x)}{\log x}.$$

The consecutive primes problem should be easier! Twin primes are rare among primes; the patterns we want should be likely to occur.

First steps towards consecutive primes

Corollary of Dirichlet's Theorem

Assume that $\phi(q) = 2$ and that a, b mod q with $a \neq b$. Then $\pi(x; q, (a, b))$ approaches infinity as $x \to \infty$.

For example, there are infinitely many primes that are 1 mod 4 such that the next prime is 3 mod 4. This is because there are infinitely many primes of both types, so there have to be infinitely many swaps.

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Theorem (Shiu (2000), Banks–Freiberg–Turnage-Butterbaugh (2015))

For any integer $r \geqslant 1$, for any $a \bmod q$,

$$\pi(x; q, \underbrace{(a, \ldots, a)}_{r \text{ times}}) \to \infty.$$

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Maynard (2016) showed further that the constant pattern of any length is likely to occur; these primes represent a positive proportion of prime numbers.
All other cases are open. For example, we cannot show that there are infinitely many primes ending in 7 such that the next prime ends in 1 in base 10.

Nearby primes as consecutive primes

The proof comes from the idea that consecutive primes should be close together, so that the result follows from Maynard's work on nearby primes.

Theorem (Maynard, 2015)

For any integer $r \geqslant 1$, for large enough k, for any "admissible" tuple of linear forms $L_1(n) = qn + a_1$, $L_2(n) = qn + a_2, \ldots, L_k(n) = qn + a_k$, for infinitely many n, at least r of the values $L_i(n)$ are prime.

$qn + a_1$ $\quad$ $qn + a_2$ $\quad$ $qn + a_3$ $\quad$ $qn + a_4$ $\quad$ $qn + a_5$

$\downarrow$

$qn + a_2$ $\quad$ $qn + a_4$ $\quad$ prime for infinitely many n

From Maynard's theorem to consecutive primes

Say we have a tuple $qn + a_1$, $qn + a_2$, $qn + a_3$, with numbers appearing in the gaps in between. Goal: for specific $t \neq a_i$, ensure that $qn + t$ is not prime.

$$\begin{array}{c} qn + a_3 \\ \vdots \\ qn + a_2 \\ \vdots \\ qn + t \\ \vdots \\ qn + a_1 + 1 \\ qn + a_1 \end{array}$$

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- Pick a prime p_t such that $p_t \nmid q$ and $t \not\equiv a_i \mod p_t$ for all i.
- Define $A_t \mod p_t$ such that $qA_t + t \equiv 0 \mod p_t$.

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$$\begin{array}{c}
qp_t n + qA_t + a_3 \\
\vdots \\
qp_t n + qA_t + a_2 \\
\vdots \\
qp_t n + qA_t + t \longleftarrow \text{not prime (divisible by } p_t\text{)} \\
\vdots \\
qp_t n + qA_t + a_1 + 1 \\
qp_t n + qA_t + a_1
\end{array}$$

- Pick a prime p_t such that $p_t \nmid q$ and $t \not\equiv a_i \mod p_t$ for all i.
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- Replace each linear form $qn + a_i$ by $qp_t n + qA_t + a_i$.

From Maynard's theorem to consecutive primes

Say we have a tuple $qn + a_1$, $qn + a_2$, $qn + a_3$, with numbers appearing in the gaps in between. Goal: for all $t \neq a_i$, ensure that $qn + t$ is not prime.

$$
\begin{array}{ll}
qQn + qA + a_3 & \\
\vdots & \text{not prime} \\
qQn + qA + a_2 & \text{not prime} \\
\vdots & \\
qQn + qA + t & \text{not prime (divisible by } p_t\text{)} \\
\vdots & \text{not prime} \\
qQn + qA + a_1 + 1 & \text{not prime} \\
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\end{array}
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- Pick a prime p_t such that $p_t \nmid q$ and $t \not\equiv a_i \mod p_t$ for all i.
- Define $A_t \bmod p_t$ such that $qA_t + t \equiv 0 \mod p_t$.
- Replace each linear form $qn + a_i$ by $qp_t n + qA_t + a_i$.
- Repeat for all t until the gaps contain *no* primes.

Constant subsequences

Maynard's theorem guarantees that *some* subtuple of size r will be simultaneously prime infinitely often.

$qn + a_1 \qquad qn + a_2 \qquad qn + a_3 \qquad qn + a_4 \qquad qn + a_5$

$$\downarrow$$

$qn + a_2 \qquad qn + a_4$ prime infinitely often

We have no control over *which* subtuple is simultaneously prime. But with a constant sequence this doesn't matter: any subsequence of a constant sequence is constant.

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$\equiv a$ $\quad$ $\equiv a$ $\quad$ prime infinitely often

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The problem with non-constant sequences

If we're looking for primes in *different* congruence classes mod q, the fact that we have no control over which subtuple is simultaneously prime becomes a problem.

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Sums of two squares: a success story

In a different setting, this technique *has* been successful. Consider the set

$$\mathbb{E} := \{n \in \mathbb{N} : n = a^2 + b^2, a, b \in \mathbb{N}\} = (E_k)_{k \in \mathbb{N}}$$

of integers that can be written as a sum of two squares, where $E_k < E_{k+1}$. By a theorem of Fermat, $n \in \mathbb{E}$ if and only if each $p|n$ with $p \equiv 3 \mod 4$ divides n to an even power. Sums of two squares often act similarly to prime numbers, and similar proofs apply in the two settings.

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Theorem (Kimmel–K., (2023))

Let q be odd and squarefree. For any a, b mod q, for any integers r_a and r_b, the pattern (a, b) appears infinitely often in the sequence $(E_k \mod q)_{k \in \mathbb{N}}$. Furthermore, the pattern

$$(\underbrace{a, \ldots, a}_{r_a \text{ times}}, \underbrace{b, \ldots, b}_{r_b \text{ times}})$$

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This theorem uses work of McGrath (2020) on a "second moment" version of Maynard's result which allows for some control over the subtuple containing infinitely many simultaneous primes. Partial progress in the prime case has been done by Banks–Freiberg–Maynard (2019), Pintz (2019), and Merikoski (2020).

Backing up: data on consecutive primes

Consider the first 10^8 primes. In 2016, Lemke Oliver and Soundararajan computed the following data for consecutive primes in arithmetic progressions (modulo 10):

a	b	Number of primes	a	b	Number of primes
1	1	4623042	7	1	6373981
	3	7429438		3	6755195
	7	7504612		7	4439355
	9	5442345		9	7431870
3	1	6010982	9	1	7991431
	3	4442562		3	6372941
	7	7043695		7	6012739
	9	7502896		9	4622916

Our conjecture from the beginning predicted that every number in this table should be ≈ 6250000.

A more precise conjecture

Conjecture (Lemke Oliver–Soundararajan, 2016)

Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a tuple of congruence classes modulo q. For certain constants $c_1(q; \mathbf{a})$ and $c_2(q; \mathbf{a})$,

$$\pi(x; q, \mathbf{a}) = \frac{\pi(x)}{\phi(q)^r} \left(1 + c_1(q; \mathbf{a}) \frac{\log \log x}{\log x} + c_2(q; \mathbf{a}) \frac{1}{\log x} + O((\log x)^{-7/4}) \right).$$

The constant $c_1(q; \mathbf{a})$ is smaller the more repeats $\mathbf{a}$ has. A version of this conjecture for sums of two squares was formulated by David–Devin–Nam–Schlitt (2021).

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$$\pi(x; q, (a, b)) = \sum_{\substack{h \leqslant x \\ h \equiv (b-a) \bmod q}} \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \mathbf{1}_{\text{prime}}(n)\mathbf{1}_{\text{prime}}(n+h) \prod_{0<t<h} (1 - \mathbf{1}_{\text{prime}}(n+t))$$

Expanding the product gives us many different sums (with different signs!) of the form

$$\sum_{\substack{t_1,\ldots,t_k \in [1,h-1] \\ t_i \text{ distinct}}} \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \mathbf{1}_{\text{prime}}(n)\mathbf{1}_{\text{prime}}(n+t_1)\mathbf{1}_{\text{prime}}(n+t_2)\cdots\mathbf{1}_{\text{prime}}(n+t_k)\mathbf{1}_{\text{prime}}(n+h).$$

These sums we need to understand *really* well.

Prime constellations: the Hardy–Littlewood k-tuples conjecture

Hardy–Littlewood Conjecture

Let $\mathcal{D} = \{d_1, \ldots, d_k\}$ be a sequence of distinct integers. As $x \to \infty$,

$$\sum_{n \leqslant x} \prod_{i=1}^{k} \mathbf{1}_{\text{prime}}(n + d_i) = \mathfrak{S}(\mathcal{D}) \frac{x}{(\log x)^k} + o(x/(\log x)^k)$$

where

$$\mathfrak{S}(\mathcal{D}) = \prod_p \frac{1 - \nu_{\mathcal{D}}(p)/p}{(1 - 1/p)^k}$$

for $\nu_{\mathcal{D}}(p)$ is the number of equivalence classes $\bmod p$ occupied by $\mathcal{D}$. $\mathfrak{S}(\mathcal{D})$ is called the *singular series* at the set $\mathcal{D}$.

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If they are independently chosen, the probability that none of $n_1, \ldots, n_k$ are 0 mod p is $\left(1 - \frac{1}{p}\right)^k$.

The probability that none of $n + d_1, \ldots, n + d_k$ are 0 mod p is $1 - \frac{\nu_{\mathcal{D}}(p)}{p}$.

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When $\mathcal{D} = \{0, 2\}$, Hardy–Littlewood predicts the asymptotic number of twin primes, via

$$\sum_{n \leqslant x} \mathbf{1}_{\text{prime}}(n) \mathbf{1}_{\text{prime}}(n+2) \sim 2 \prod_{p \geqslant 3} \frac{1 - 2/p}{(1 - 1/p)^2} \frac{x}{(\log x)^2} \approx 1.32 \frac{x}{(\log x)^2}.$$

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When $\mathcal{D} = \{0, 1\}$, $\mathfrak{S}(\mathcal{D}) = 0$, since the factor at $p = 2$ is $\frac{1-2/2}{(1-1/2)^2} = 0$. "Either n or $n + 1$ is even, so there are very few consecutive primes."

Tuples via Hardy–Littlewood

$$\sum_{\substack{t_i \in [1, h-1] \\ t_i \text{ distinct}}} \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \mathbf{1}_{\text{prime}}(n) \mathbf{1}_{\text{prime}}(n + t_1) \mathbf{1}_{\text{prime}}(n + t_2) \cdots \mathbf{1}_{\text{prime}}(n + t_k) \mathbf{1}_{\text{prime}}(n + h)$$

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$$\sim \sum_{\substack{t_1,\ldots,t_k \in [1,h-1] \\ t_i \text{ distinct}}} \mathfrak{S}(\{0, t_1, \ldots, t_k, h\}) \frac{x}{(\log x)^k}$$

$$= \frac{x}{(\log x)^k} \sum_{\substack{t_1,\ldots,t_k \in [1,h-1] \\ t_i \text{ distinct}}} \mathfrak{S}(\{0, t_1, \ldots, t_k, h\}).$$

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What really needs to be understood for these (and other!) computations is sums of the constants $\mathfrak{S}(\mathcal{D})$ as the tuple $\mathcal{D}$ changes.

Results on sums of singular series

- Gallagher (1976): $\mathfrak{S}(\mathcal{D})$ is 1 when averaged over k-tuples $\mathcal{D} \subseteq [1, h]$ (and consequences for the number of intervals of width $\log x$ containing exactly k primes)

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- Montgomery (1971), Goldston (1989): sharper estimates of $\sum_{n \leqslant h} \mathfrak{S}(\{0, n\})$; Goldston used this to find sums of two primes in short intervals
- Montgomery–Soundararajan (2004): estimates of $R_k(h)$, a sort of error term for Gallagher's average, when k is even and provide upper bounds when k is odd (and consequences for the distribution of primes in longer intervals)
- K. (2024) and Bloom–K. (2024+): nearly tight upper bounds on $R_k(h)$ when k is odd

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- Open question: Show that HL implies the Lemke Oliver–Soundararajan conjecture.

Much is left to be understood about consecutive primes.

Thank you!