Fock covariance for product systems and the Hao–Ng isomorphism problem

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Table of contents

Product Systems and their representations

- In this talk P is a unital subsemigroup of a discrete group G.
- We say that a family $X = \{X_p\}_{p \in P}$ of closed operator spaces in a common $\mathcal{B}(H)$ is a *product system* if the following are satisfied:
  1. $A := X_e$ is a C*-algebra;
  2. $X_p \cdot X_q \subseteq X_{pq}$ for all $p, q \in P$;
  3. $X_p^* \cdot X_{pq} \subseteq X_q$ for all $p, q \in P$.
- It follows that each X_p has a natural Hilbert A-module structure.
- A *representation* $t = \{t_p\}_{p \in P}$ of X consists of a family of linear maps $t_p \colon X_p \to \mathcal{B}(K)$ such that:
  1. t_e is a $*$-representation of $A := X_e$;
  2. $t_p(\xi_p) t_q(\xi_q) = t_{pq}(\xi_p \xi_q)$ for all $\xi_p \in X_p$ and $\xi_q \in X_q$;
  3. $t_p(\xi_p)^* t_{pq}(\xi_{pq}) = t_q(\xi_p^* \xi_{pq})$ for all $\xi_p \in X_p$ and $\xi_{pq} \in X_{pq}$.
- We say that t is *injective* if t_e is injective.
- We use the notation

$$t^{(p)} \colon \mathcal{K}(X_p) := [X_p X_p^*] \to \mathcal{B}(K); \xi_p \eta_p^* \mapsto t_p(\xi_p) t_p(\eta_p)^*,$$

for the induced $*$-representation of the compact operators $\mathcal{K}(X_p)$.

Product Systems and their representations

- For a set $Z \subseteq P$ and $p \in P$ we write

$$pZ := \{px \mid x \in Z\} \quad \text{and} \quad p^{-1}Z := \{y \in P \mid py \in Z\}.$$

- We set $\mathcal{J} := \{q_n^{-1}p_n \dots q_1^{-1}p_1 P \mid n \in \mathbb{N}; p_i, q_i \in P\} \cup \{\emptyset\}$. It follows that $\mathcal{J}$ is $\cap$-closed.
- We will write $\mathbf{x}, \mathbf{y}$ etc. for the elements in $\mathcal{J}$ and we will refer to them as *constructible ideals*.
- Consider the *Fock space* $\mathcal{F}X := \sum_{r \in P}^{\oplus} X_r$ as a Hilbert A-module and the *Fock representation* $\lambda \colon X \to \mathcal{L}(\mathcal{F}X)$ where

$$\lambda_p(\xi_p)\eta_r = \xi_p \cdot \eta_r \quad \text{and} \quad \lambda_p(\xi_p)^*\eta_r = \begin{cases} \xi_p^* \cdot \eta_r & \text{if } r \in pP, \\ 0 & \text{if } r \notin pP. \end{cases}$$

- We set $\mathcal{T}_\lambda(X) := \mathrm{C}^*(\lambda)$ and $\mathcal{T}(X) := \mathrm{C}^*(\hat{t})$ where $\hat{t}$ is the universal representation of X.

Product Systems and their representations

- t will be called *equivariant* if $\mathrm{C}^*(t)$ admits a *coaction* i.e. a $*$-homomorphism

$$\delta\colon \mathrm{C}^*(t) \to \mathrm{C}^*(t) \otimes \mathrm{C}^*_{\max}(G); t_p(\xi_p) \mapsto t_p(\xi_p) \otimes u_p,$$

  where u_p are the generators of the universal C*-algebra of G.
- For $\mathbf{x} \in \mathcal{J}$ we define $\mathbf{K}_{\mathbf{x},t_*}$ to be the closed linear span of the spaces

$$\left(t_{p_1}(X_{p_1})^*\right)^{\varepsilon} t_{q_1}(X_{q_1}) \cdots t_{p_n}(X_{p_n})^* t_{q_n}(X_{q_n})^{\varepsilon'}$$

  for any $p_1, q_1, \dots, p_n, q_n \in P$ and $\varepsilon, \varepsilon' \in \{0,1\}$ that satisfy

$$p_1^{-\varepsilon} q_1 \cdots p_n^{-1} q_n^{\varepsilon'} = e \text{ and } q_n^{-\varepsilon'} p_n \dots q_1^{-1} p_1^{\varepsilon} P = \mathbf{x}.$$

- We set $[\mathrm{C}^*(t)]_g := \{c \in \mathrm{C}^*(t) \mid \delta(c) = c \otimes u_g\}$ for each $g \in G$.
- There is an induced conditional expectation $E\colon \mathrm{C}^*(t) \to [\mathrm{C}^*(t)]_e$. We say that t *admits a normal coaction by* G if E is faithful.
- $\hat{t}$ is equivariant and λ admits a normal coaction by G.

Fock covariant representations

We say that t is a *Fock covariant* representation of X if

$$\ker \lambda_* \cap [\mathcal{T}(X)]_e \subseteq \ker t_* \cap [\mathcal{T}(X)]_e,$$

where $\lambda_* : \mathcal{T}(X) \to \mathcal{T}_\lambda(X)$ and $t_* : \mathcal{T}(X) \to \mathrm{C}^*(t)$ are the induced $*$-epimorphisms by the universality of $\mathcal{T}(X)$.

Theorem (Kakariadis–P. 2024)

An equivariant injective representation t of X is Fock covariant if and only if t satisfies the following conditions:

1. $\mathbf{K}_{\emptyset, t_*} = (0)$.
2. *For any $\cap$-closed $\mathcal{F} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq \mathcal{J}$ such that $\bigcup_{i=1}^n \mathbf{x}_i \neq \emptyset$, and any $b_{\mathbf{x}_i} \in \mathbf{K}_{\mathbf{x}_i, \hat{t}_*}$, with $i = 1, \dots, n$, the following property holds:*

$$\text{if } \sum_{i : r \in \mathbf{x}_i} t_*(b_{\mathbf{x}_i}) t_r(X_r) = (0) \text{ for all } r \in \bigcup_{i=1}^n \mathbf{x}_i, \text{ then}$$

$$\sum_{i=1}^n t_*(b_{\mathbf{x}_i}) = 0.$$

Fock covariant representations

- P is said to be a *right LCM semigroup* if $\mathcal{J} = \{pP \mid p \in P\} \bigcup \{\emptyset\}$.
- A p.s. X over a right LCM semigroup P is *compactly aligned* if

$$\lambda^{(p)}(\mathcal{K}(X_p))\lambda^{(q)}(\mathcal{K}(X_q)) \subseteq \lambda^{(w)}(\mathcal{K}(X_w)) \text{ when } pP \cap qP = wP.$$

- We say that a representation t of X is *Nica covariant* if

$$t^{(p)}(\mathcal{K}(X_p))t^{(q)}(\mathcal{K}(X_q)) \subseteq \begin{cases} t^{(w)}(\mathcal{K}(X_w))) & \text{if } pP \cap qP = wP, \\ (0) & \text{if } pP \cap qP = \emptyset. \end{cases}$$

Using our characterisation we give an alternative proof of:

Proposition (Dor-On–Kakariadis–Katsoulis–Laca–Li 2020)

A representation of X is Fock covariant if and only if it is Nica covariant.

Fock covariant representations

- $P \subseteq G$ is a unital subsemigroup and $X_p = \mathbb{C}$ for each $p \in P$.
- $\alpha = (p_1, q_1, \ldots, p_n, q_n)$ is called *neutral* if $p_1^{-1} q_1 \cdots p_n^{-1} q_n = e$.
- For a unital (i.e. $t_e(1) = 1$) representation t of X set $w_p := t_p(1)$.
- For a neutral word $\alpha = (p_1, q_1, \ldots, p_n, q_n)$ we write

$$K(\alpha) := q_n^{-1} p_n \ldots q_1^{-1} p_1 P \text{ and } \dot{w}_\alpha := w_{p_1}^* w_{q_1} \cdots w_{p_n}^* w_{q_n}.$$

Using our characterisation we give an alternative proof of:

Proposition (Laca–Sehnem 2021)

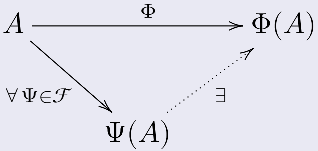
A unital (equivariant) representation t is Fock covariant if and only if

1. $w_e = 1$,
2. $\dot{w}_\alpha = 0$ *if* $K(\alpha) = \emptyset$ *for a neutral word* α, *and*
3. $\dot{w}_\alpha = \dot{w}_\beta$ *if* $K(\alpha) = K(\beta)$ *for neutral words* α *and* β,
4. $\prod_{\beta \in F} (\dot{w}_\alpha - \dot{w}_\beta) = 0$ *whenever* α *is a neutral word,* F *is a finite set of neutral words and* $K(\alpha) = \bigcup_{\beta \in F} K(\beta)$.

Fock covariant representations

Definition

Let $\mathcal{F}$ be a class of $*$-representations of a C*-algebra A. We say that a Φ in $\mathcal{F}$ is $\mathcal{F}$*-boundary* and denote it by $\partial\mathcal{F}$ if:



Example

Let A, B be C*-algebras and

$$\mathcal{F}^{\odot} := \{\Psi\colon A \otimes_{\max} B \to \mathcal{B}(H) \mid \Psi \text{ is injective on } A \odot B\}.$$

Then the $*$-homomorphism $A \otimes_{\max} B \to A \otimes B$ is $\mathcal{F}^{\odot}$-boundary.

Fock covariant representations

Theorem (Exel 1997)

Let δ be a coaction on a C*-algebra A and let $\mathcal{A} := \{\mathcal{A}_g\}_{g\in G}$ to be the induced Fell bundle. Let

$$\mathcal{F}_{\mathcal{A}^\delta} := \{\text{ equivariant repn's of } \mathrm{C}^*_{\max}(\mathcal{A}) \text{ that are injective on } \mathcal{A}_e\}.$$

Then the canonical $*$-epimorphism $\mathrm{C}^*_{\max}(\mathcal{A}) \to \mathrm{C}^*_{\lambda}(\mathcal{A})$ is $\mathcal{F}_{\mathcal{A}^\delta}$-bdy.

Theorem (Hamana 1978, Dritschel–McCullough 2000)

Let $\mathfrak{A}$ be an operator algebra and let

$$\mathcal{F}_{\mathfrak{A}} := \{\Psi\colon \mathrm{C}^*_{\max}(\mathfrak{A}) \to \mathcal{B}(K) \mid \Psi|_{\mathfrak{A}} \text{ is c.is.}\}.$$

Then $\mathcal{F}_{\mathfrak{A}}$ admits an $\mathcal{F}_{\mathfrak{A}}$-boundary repn, called the C*-envelope of $\mathfrak{A}$, denoted by $\mathrm{C}^*_{\mathrm{env}}(\mathfrak{A})$.

Fock covariant representations

Let X be a product system over P. A key question in the theory of product systems was whether there exists a boundary representation for

$$\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} := \{ \text{ equivariant Fock covariant injective representations of } X \ \},$$

and what is its relation to the C*-envelope of $\mathcal{T}_\lambda(X)^+ := \overline{\mathrm{alg}}(\lambda_p(X_p))$?

Affirmative answers

- For $P = \mathbb{Z}_+$ Katsoulis–Kribs (2004) showed that $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+) \simeq \mathcal{O}_X$ and Katsura (2007) showed that $\mathcal{O}_X \simeq \partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X}$.
- For (G, P) quasi-lattice and with assumptions on X Carlsen–Larsen–Sims–Vittadello (2011) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathcal{N}\mathcal{O}^r_X$.
- For $P = \mathbb{Z}^d_+$ and X arising from dynamics Davidson–Fuller–Kakariadis (2014) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+)$.
- For (G, P) an abelian lattice Dor-On–Katsoulis (2018) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+)$.

Fock covariant representations

Affirmative answers

- For P a right LCM Dor-On–Kakariadis–Katsoulis–Laca–Li (2020) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \delta)$.

- For any P and X trivial Kakariadis–Katsoulis–Laca–Li (2021) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \delta)$.

- For general product systems Sehnem (2021) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+)$.

Fock covariant representations

Strong covariant relations (Sehnem 2018)

(1) Sehnem (2018) constructed a universal C*-algebra $A \times_X P$ of a product system X with the following property:

> *$X \hookrightarrow A \times_X P$ and if $t\colon A \times_X P \to \mathcal{B}(H)$ is a $*$-representation with $t|_A$ injective then t is injective on the fixed point algebra of $A \times_X P$.*

(2) $A \times_X P$ admits a coaction by G and a Fell bundle, say $\mathcal{SC}_G X$. We then write

$$\mathrm{C}^*_{\max}(\mathcal{SC}_G X) = A \times_X P \text{ and } \mathrm{C}^*_\lambda(\mathcal{SC}_G X) = A \times_{X,\lambda} P.$$

Theorem (Sehnem 2021)

$$\partial \mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+) \simeq A \times_{X,\lambda} P. \tag{1}$$

The reduced Hao–Ng isomorphism problem

- Let $\mathfrak{H}$ be a locally compact group.
- A *generalised gauge action* is an action of $\mathfrak{H}$ on $\mathcal{T}_\lambda(X)$ such that

$$\alpha_{\mathfrak{h}}(\lambda_p(X_p)) = \lambda_p(X_p) \text{ for all } p \in P \text{ and } \mathfrak{h} \in \mathfrak{H}.$$

- There is an induced product system $X \rtimes_{\alpha,\lambda} \mathfrak{H} := \{X_p \rtimes_{\alpha,\lambda} \mathfrak{H}\}_{p \in P}$.
- In the case where $\mathfrak{H}$ is discrete we have

$$X_p \rtimes_{\alpha,\lambda} \mathfrak{H} = \overline{\operatorname{span}}\{\overline{\pi}(\lambda_p(\xi_p))U_{\mathfrak{h}} \mid \xi_p \in X_p, \mathfrak{h} \in \mathfrak{H}\},$$

where $\overline{\pi} \times U$ is a faithful representation of $\mathcal{T}_\lambda(X) \rtimes_{\alpha,\lambda} \mathfrak{H}$.

The reduced Hao–Ng isomorphism problem

Is α compatible with the reduced strong covariant functor, i.e., is there a canonical $*$-isomorphism

$$A \times_{X \rtimes_{\alpha,\lambda} \mathfrak{H}, \lambda} P \overset{?}{\simeq} (A \times_{X,\lambda} P) \rtimes_{\alpha,\lambda} \mathfrak{H}.$$

The reduced Hao–Ng isomorphism problem

Affirmative answers

- For $P = \mathbb{Z}_+$ and $\mathfrak{H}$ l. c. amenable (Hao–Ng 2008).
- For $P = \mathbb{Z}_+$ and $\mathfrak{H}$ discrete (Katsoulis 2017).
- For (G, P) an abelian lattice and $\mathfrak{H}$ discrete (Dor-On–Katsoulis 2018).
- For (G, P) an abelian lattice and $\mathfrak{H}$ l. c. abelian (Katsoulis 2020).
- For P a right LCM and $\mathfrak{H}$ discrete (Dor-On–Kakariadis– Katsoulis–Laca–Li 2020).

We should note that in all the cases above X is considered to be non-degenerate i.e., $[A \cdot X_p] = X_p$ for every $p \in P$.

The reduced Hao–Ng isomorphism problem

Definition (Katsoulis–Ramsey 2019)

Let $\mathfrak{A}$ be an operator algebra and α an action of a locally compact group $\mathfrak{H}$ on $\mathfrak{A}$ by completely isometric isomorphisms. The *(nonselfadjoint) reduced crossed product* $\mathfrak{A} \rtimes_{\alpha,\lambda} \mathfrak{H}$ is the norm-closed sublagebra generated by the copies of $\mathfrak{A}$ and $\mathfrak{H}$ inside $\mathrm{C}^*_{\mathrm{env}}(\mathfrak{A}) \rtimes_{\dot{\alpha},\lambda} \mathfrak{H}$ where $\dot{\alpha}$ is the induced action on $\mathrm{C}^*_{\mathrm{env}}(\mathfrak{A})$.

The next proposition, in the case $\mathfrak{A}$ admits a contractive approximate identity, was proved by Katsoulis (2017) when $\mathfrak{H}$ is discrete and by Katsoulis–Ramsey (2019) when $\mathfrak{H}$ is abelian. By using maximal representations we can remove the c.a.i. hypothesis, when $\mathfrak{H}$ is discrete.

Proposition

Let $\mathfrak{H}$ be a discrete group acting by α on an operator algebra $\mathfrak{A}$. Then

$$\mathrm{C}^*_{\mathrm{env}}(\mathfrak{A} \rtimes_{\alpha,\lambda} \mathfrak{H}) \simeq \mathrm{C}^*_{\mathrm{env}}(\mathfrak{A}) \rtimes_{\dot{\alpha},\lambda} \mathfrak{H}. \tag{2}$$

The reduced Hao–Ng isomorphism problem

Let $\mathfrak{H}$ be discrete. Using our characterisation of equivariant Fock covariant injective representations we obtain the following:

Proposition

The identity representation $\iota\colon X \rtimes_{\alpha,\lambda} \mathfrak{H} \to \mathcal{T}_\lambda(X) \rtimes_{\alpha,\lambda} \mathfrak{H}$ is a Fock covariant injective representation that admits a normal coaction.

We also need the following corollary of a result of Sehnem (2021):

Corollary

If t is a Fock covariant injective representation of X that admits a normal coaction, then the map

$$\mathcal{T}_\lambda(X)^+ \to \overline{\mathrm{alg}}\{t_p(X_p) \mid p \in P\}; \lambda_p(\xi_p) \mapsto t_p(\xi_p),$$

is a (well defined) completely isometric isomorphism.

The reduced Hao–Ng isomorphism problem

Combining the preceding propositions we obtain

$$\mathcal{T}_\lambda(X \rtimes_{\alpha,\lambda} \mathfrak{H})^+ \simeq \mathcal{T}_\lambda(X)^+ \rtimes_{\alpha,\lambda} \mathfrak{H},$$

then (2) implies that

$$\mathrm{C}^*_{\text{env}}(\mathcal{T}_\lambda(X \rtimes_{\alpha,\lambda} \mathfrak{H})^+) \simeq \mathrm{C}^*_{\text{env}}(\mathcal{T}_\lambda(X)^+) \rtimes_{\alpha,\lambda} \mathfrak{H},$$

and (1) finishes the proof.
Hence the reduced Hao–Ng isomorphism problem for generalised gauge actions by discrete groups has an affirmative answer:

Theorem (Kakariadis–P. 2024)

There exists a canonical $*$-isomorphism

$$(A \rtimes_{\alpha,\lambda} \mathfrak{H}) \times_{X \rtimes_{\alpha,\lambda} \mathfrak{H},\lambda} P \simeq (A \times_{X,\lambda} P) \rtimes_{\dot{\alpha},\lambda} \mathfrak{H}.$$

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