Hilbert transforms and Cotlar-type identities for groups acting on tree-like structures

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The Hilbert transform

Definition

For $f \in C_c^\infty(\mathbb{R})$,

$$(Hf)(x) = \mathrm{p.v.} \int_{\mathbb{R}} f(y) \frac{1}{x-y} dy.$$

As a Fourier multiplier:

$$\widehat{(Hf)}(\xi) = -i \, \mathrm{sgn}(\xi) \cdot \widehat{f}(\xi), \quad \xi \in \mathbb{R}.$$

Question

Is H well-defined and bounded on $L_p(\mathbb{R})$ $(p \neq 2)$?

The boundedness of Hilbert transform on $\mathbb{R}$

Results:

- Unbounded on L_p for $p = 1, \infty$.
- Trivially bounded on L_2.
- **M. Riesz (1924)** Bounded for $1 < p < \infty$.
- **Cotlar (1955)** Recursive: $p = 2^k +$ Marcinkiewicz's Interpolation.
- **Kolmogorov (1924)** Bounded from L_1 to weak L_1.

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Classical Cotlar Identity:

$$(Hf)^2 = f^2 + 2H(f\,Hf).$$

Generalised by Mei and Ricard (2017) for amalgamated free product of von Neumann algebras.

Motivation:

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k\theta} \longrightarrow f$$

in L_p-norm, for any $f \in L_p(\mathbb{T})$?

Problem

Let $f \in L_p(\mathbb{T})$ for $1 < p < \infty$. Do we have

$$\sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{2\pi i k\theta} \longrightarrow f(\theta) \text{ in } L_p\text{-norm?}$$

Motivation:

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\widehat{f}(k)e^{2\pi ik\theta}\longrightarrow f$$

in L_p-norm, for any $f\in L_p(\mathbb{T})$?

Problem

Let $f\in L_p(\mathbb{T})$ for $1<p<\infty$. Do we have

$$\lim_{N\to\infty}(T_{\mathbf{1}_{[-N,N]}}f)(\theta)=\lim_{N\to\infty}\sum_{k=-N}^{N}\widehat{f}(k)e^{2\pi ik\theta}\longrightarrow f(\theta)\text{ in }L_p\text{-norm?}$$

$$L_p\text{-norm convergence}\iff\sup_N\|T_{\mathbf{1}_{[-N,N]}}:L_p(\mathbb{T})\to L_p(\mathbb{T})\|<\infty.$$

Symbol	Multiplier
$i\,\mathrm{sgn}(k)$	H
$\mathbf{1}_{[0,\infty)}(k)$	$\frac{1}{2}(\mathbf{1}+iH)$
$\mathbf{1}_{[a,\infty)}(k)$	$\underbrace{\frac{1}{2}(\mathbf{1}+iM_{e^{-2\pi iax}}HM_{e^{2\pi iax}})}_{H_a}$
$\mathbf{1}_{[a,b]}(k)=\mathbf{1}_{[a,\infty)}(k)\cdot\mathbf{1}_{[-b,\infty)}(-k)$	$H_a\tilde{H}_b$

where $M_{f(x)}g(x)=f(x)g(x)$.

$$\|H:L_p(\mathbb{T})\to L_p(\mathbb{T})\|<\infty \iff \sup_N \|T_{\mathbf{1}_{[-N,N]}}:L_p(\mathbb{T})\to L_p(\mathbb{T})\|<\infty.$$

$\rightsquigarrow$ Connections to CBAP (completely bounded approximation property).

Non-Abelian groups

G: discrete group.

Left regular representation

$\lambda : \mathrm{G} \to \mathcal{U}(\ell_2(\mathrm{G}))$ with $\lambda_g \varphi(h) = \varphi(g^{-1}h)$. $\mathcal{L}(\mathrm{G}) = \{\lambda_g\}''_{g \in \mathrm{G}}$.

Non-abelian Fourier transform

For $\widehat{f} \in \ell_1(\mathrm{G}), f := \sum_{\mathrm{G}} \widehat{f}(g)\lambda_g$ is a bounded linear map $\ell_2(\mathrm{G}) \to \ell_2(\mathrm{G})$.

Non-commutative L_p-spaces

$L_p(\widehat{\mathrm{G}}) := L_p(\mathcal{L}(\mathrm{G}), \tau) =$ "$\{f : \tau(|f|^p)^{\frac{1}{p}} < \infty\}$" with $\tau(f) = \widehat{f}(e)$.

Fourier multipliers on $\mathcal{L}(\mathrm{G})$:

$m \in \ell_\infty(\mathrm{G}) \rightsquigarrow T_m f := \sum_{\mathrm{G}} m(g)\widehat{f}(g)\lambda_g$.

Noncommutative Cotlar identity

Problem: Does it hold that $\|H = T_m : L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}})\| < \infty$?

Let G be a discrete group and Γ be a subgroup of G. There exists a conditional expectation E from $\mathcal{L}(\mathrm{G})$ to $\mathcal{L}(\Gamma)$.

Generalised nc Cotlar identity (Mei-Ricard)

$$H(f)H(f)^* - E\big[H(f)H(f)^* - H(fH(f)^*) - H(fH(f)^*)^* + H(H(ff^*))^*)\big]$$
$$= H(fH(f)^*) + H(fH(f)^*)^* - H(H(ff^*))^*).$$

Proposition (González Pérez-Parcet-X)

Let $H = T_m$. TFAE:

- Generalised nc Cotlar identity holds for H.
- Condition on the symbol:

$$(m(gh) - m(g))(\overline{m(g^{-1})} - \overline{m(h)}) = 0, \forall g \in G - \Gamma, \forall h \in \mathrm{G}.$$

Proof.

Let $\widetilde{x} = x - E(x)$, $\forall x \in \mathcal{L}(\mathrm{G})$. Then the generalised nc Cotlar identity

$$H(\widetilde{f) H(f})^* - H(\widetilde{f H(f})^*) - H(\widetilde{f H(f})^*)^* + H(\widetilde{H(ff^*})))^* = 0$$

for $f \in \mathbb{C}[G]$ is equivalent to

Proof.

Let $\widetilde{x} = x - E(x)$, $\forall x \in \mathcal{L}(\mathrm{G})$. Then the generalised nc Cotlar identity

$$\widetilde{H(f)H(f)^*} - H(\widetilde{fH(f)^*}) - H(\widetilde{fH(f)^*})^* + H(\widetilde{H(ff^*)})^* = 0$$

for $f \in \mathbb{C}[G]$ is equivalent to

$$\sum_{g \in G - \Gamma} \sum_{h \in G} \left[m(gh)\overline{m(h)} - m(g)\overline{m(h)} - \overline{m(g^{-1})}m(gh) + m(g)\overline{m(g^{-1})} \right] \widehat{f}(gh)\overline{\widehat{f}(h)}\lambda_g = 0.$$

This is equivalent to

$$m(gh)\overline{m(h)} - m(g)\overline{m(h)} - \overline{m(g^{-1})}m(gh) + m(g)\overline{m(g^{-1})} = 0,$$

for any $g \in G - \Gamma$ and $h \in \mathrm{G}$, i.e.

$$(m(gh) - m(g))(\overline{m(g^{-1})} - \overline{m(h)}) = 0, \ \forall g \in G - \Gamma, \forall h \in \mathrm{G}.$$

□

Lemma (González Pérez-Parcet-X)

Let G *be a discrete group and* Γ *be a subgroup of* G*. Let* E *be the conditional expectation from* $\mathcal{L}(\mathrm{G})$ *to* $\mathcal{L}(\Gamma)$*. If the symbol* m *is* **left** Γ**-invariant***, then we have for any* $p \geq 1$

$$\max\{\|E[H(f)H(f)^*]\|_p, \|E[H(fH(f)^*)]\|_p, \\ \|E[H(fH(f)^*)^*]\|_p, \|E[H(H(ff^*))^*)]\|_p\} \leq \|E(ff^*)\|_p.$$

Proposition (Mei-Ricard '17, González Pérez-Parcet-X)

Let $H : L_2(\widehat{\mathrm{G}}) \to L_2(\widehat{\mathrm{G}})$ be bounded. If

$$(m(gh) - m(g))(\overline{m(g^{-1})} - \overline{m(h)}) = 0, \forall g \in G - \Gamma, \forall h \in \mathrm{G}$$

and m is left Γ-invariant, then $\|H : L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}})\| \lesssim \left(\frac{p^2}{p-1}\right)^\beta$, $\beta = \log_2(1 + \sqrt{2})$.

$\mathbb{R}$-trees

Let X be a Hausdorff topological space.

Arc γ on X: a subset of X s.t. $\exists$ an bijective continuous function $\varphi : [0, 1] \to \gamma$.

X **is uniquely arcwise connected (UAC):** if any two points in X are joined by a unique arc.

$\mathbb{R}$-**tree:** a metrisable UAC space X s.t. there is metric d such that the unique arc joining two points is isometric to a closed interval in the real line.

Examples: $\mathbb{R}$, simplicial trees.

Hilbert transforms for groups acting on $\mathbb{R}$-trees

Let G be a group acting on a UAC topological space X by homeomorphisms.
Choose a point P_0 in X and write $X \backslash \{P_0\}$ as the disjoint union of arc-connected subsets $\sqcup_i X_i = X \backslash \{P_0\}$.

Definition

For every arc-connected subset X_i, we choose a constant $C_i \in \mathbb{C}$ such that $\sup_i C_i < \infty$. Define a bounded function on X by

$$\widetilde{m}|_{X_i} \equiv C_i, \quad \text{and} \quad \widetilde{m}(P_0) = 0.$$

Then the function $\widetilde{m}$ induces a function on G by

$$m(g) = \widetilde{m}(g \cdot P_0) \quad \text{for any } g \in \mathrm{G}.$$

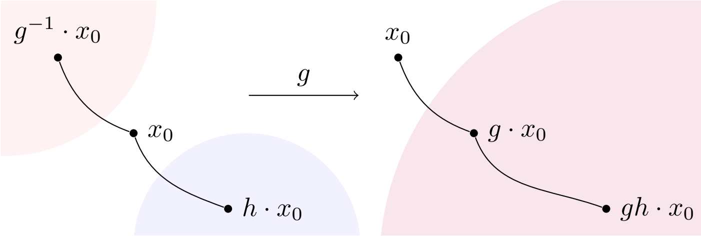
Remark

Let G_{P_0} be the stabiliser of P_0 in G. m is right G_{P_0}-invariant.

Theorem (González Pérez-Parcet-X.)

The function m defined above satisfies the Cotlar condition that for any $g, h \in \mathrm{G}$ s.t. $g \notin G_{P_0}$,

$$(m(gh) - m(g))(\overline{m(g^{-1})} - \overline{m(h)}) = 0.$$



Remark

To ensure T_m is bounded on $L_p(\widehat{\mathrm{G}})$, we need to take $C_i = C_j$ if $X_i = g \cdot X_j$ for some $g \in G_{P_0}$. In particular, when G_{P_0} is a normal subgroup of G, m is left G_{P_0}-invariant by default.

Theorem (González Pérez-Parcet-X.)

The function m defined above is both left and right G_{P_0}-invariant and satisfies the Cotlar condition that for any $g, h \in \mathrm{G}$ s.t. $g \notin G_0$,

$$(m(gh) - m(g))(\overline{m(g^{-1})} - \overline{m(h)}) = 0.$$

Remark

When G is finitely generated, T_m defined above is trivial iff G has property (F$\mathbb{R}$) (any action on an $\mathbb{R}$-tree has a global fixed point).

Theorem (González Pérez-Parcet-X.)

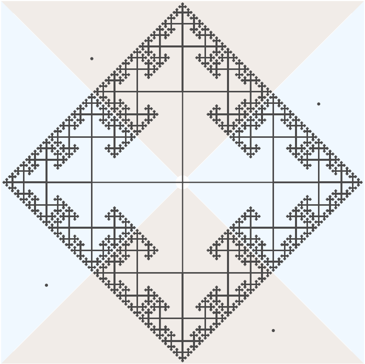
Let G act on a UAC space X. We can define a bounded function on G satisfying Cotlar, and therefore

$$\|T_m : L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}})\| \lesssim \left(\frac{p^2}{p-1}\right)^{\beta}, \beta = \log_2(1+\sqrt{2}).$$

Example I: Free groups

Consider the free group with 2 generators $\mathbb{F}_2$ acting on its Cayley graph.

$$m = C_1 \mathbb{1}_{\mathcal{W}_a} + C_2 \mathbb{1}_{\mathcal{W}_b} + C_3 \mathbb{1}_{\mathcal{W}_{a^{-1}}} + C_4 \mathbb{1}_{\mathcal{W}_{b^{-1}}}.$$



Example II: Baumslag-Solitar groups

The group presentation of $\mathrm{B}(m, n)$ is given by

$$\langle r, t : tr^m t^{-1} = r^n \rangle.$$

So $\mathrm{B}(m, n)$ is an HNN-extension based on $\Gamma = \mathbb{Z} = \langle r \rangle$ associated with $A = m\mathbb{Z}$ and $B = n\mathbb{Z}$.
Consider $\mathrm{B}(m, n)$ acting on its Bass-Serre tree by left multiplication.

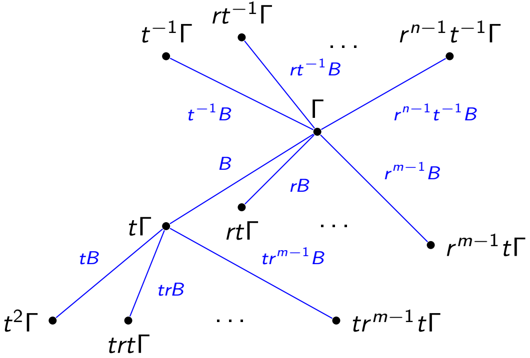


Figure: Bass-Serre tree of $\mathrm{B}(m, n)$

Each element $g \in \mathrm{B}(m, n)$ has a unique representation of the form

$$g = r^{i_0} t^{e_1} r^{i_1} \cdots t^{e_j} r^{i_j} \quad \text{with } i_0 \in \mathbb{Z}, e_1, \ldots e_j \in \{\pm 1\},$$

if $e_k = 1$, then $0 \leq i_k \leq m-1$, if $e_k = -1$, then $0 \leq i_k \leq n-1$ for $k \geq 1$.

Consider the action of $\mathrm{B}(m, n)$ on its Bass-Serre tree by left multiplication, the symbol (from our definition) of the Hilbert transform on $\mathrm{B}(m, n)$ admits the following form. For $g \notin \Gamma$,

$$m(g) = C_1, \quad \text{if } e_1 = 1$$

and

$$m(g) = C_2, \quad \text{if } e_1 = -1,$$

where C_1 and C_2 are constants. If $g \in \Gamma$, we have $m(g) = 0$.

Example III: Left-orderable groups

A group G is said to be **left-orderable** if there exists a left-invariant total order $\preceq$ on G, i.e. if $a \preceq b$ then $ga \preceq gb$ for all $g \in \mathrm{G}$.

- Torsion-free nilpotent groups, such as the Heisenberg group;
- Baumslag-Solitar groups $\mathrm{BS}(1, \mathrm{n})$ for $n \geq 2$;
- Surface groups and Thompson group F.

Lemma

Every countable left-orderable group acts on the real line $\mathbb{R}$ by orientation preserving homeomorphisms and without global fixed point.

Proposition (González Pérez-Parcet-X.)

Let G be a countable left-orderable group. If we view G as a group acting on the real line $\mathbb{R}$ and choose $P_0 = 0$ in $\mathbb{R}$, then the corresponding symbol m given by our definition will be

$$m(g) = C_1 \quad \text{if } e \preceq g \text{ and } g \neq e,$$

$$m(g) = C_2 \quad \text{if } g \preceq e \text{ and } g \neq e.$$

Recent developments and questions

- (Lu-X.): We have defined and studied the Hilbert transforms on Coxeter groups.
- **Question**: Can we find a more general geometric object to replace the " $\mathbb{R}$-trees"? For instance, buildings?