Noncommutative Field and Gauge Theory

Patrizia Vitale

Dipartimento di Fisica Università di Napoli Federico II and INFN

Developments in Modern Mathematics
Göttingen, 2-5 September 2024

Outline

- Motivations for NC spacetime
- Quantum Mechanics as the prototypical NC geometry
- The Weyl-Wigner-Moyal approach to QM

  The Moyal space-time
- Classical gauge theory of fundamental interactions
- The standard picture of NC gauge theory of fundamental interactions
  - The NC differential calculus
  - NC electrodynamics on Moyal space
  - NC electrodynamics on $\mathbb{R}^3_\lambda$
- Symmetries
- Other approaches
  - Twist
  - Noncommutative field theory from angular twist
- Open problems
- Perspectives

Motivations for NCG *of spacetime*

- Regularization of QFT in the UV regime [Heisenberg '30, Snyder '47];
- Incompatibility between Quantum Mechanics and General Relativity at small length scales [Bronstein '36, Doplicher-Fredenhagen-Roberts '94];
- Space-time discreteness emerging from different models of quantum gravity [e.g. LQG where the spectrum of area and volume operators is discrete [Ashtekhar '01];
- NC behaviour emerging in the low energy regime of string theory in the presence of a constant background field B [Seiberg-Witten '99], already in [Witten '86] in the context of string field theory;
- Connes standard model of fundamental interactions [Connes-Lott '91, Connes-Chammseddine '97]

DFR argument

Attempts to spatially localize an event with extreme precision cause gravitational collapse so that spacetime (pseudo-Riemannian manifold) below the Planck scale $\lambda_P = (\frac{G\hbar}{c^3})^1/2 \simeq 1.6 \times 10^{-33}$ has no operational meaning

- Heisenberg uncertainty principle: measuring the spacetime coordinate of a particle with great accuracy, a, causes an uncertainty in momentum of order $\frac{1}{a}$ $\Longrightarrow$ an energy of order $1/a^2$ is transmitted to the system and concentrated at some time in the localization region;
- General Relativity: the associated energy momentum tensor $T_{\mu\nu}$ generates a gravitational field solution of Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R\eta_{\mu\nu} = 8\pi T_{\mu\nu}$$

- the smaller the uncertainty Δx_μ the stronger will be the gravitational field generated
- as $\Delta x \to 0$ the field becomes so strong as to prevent light or other signals from leaving the region (what physicists call a *black hole*) $\Longrightarrow$ operational meaning can no longer be attached to the localization

By requiring that no blackhole is produced DFR infer that the Δx^μ cannot be made simultaneously arbitrary small
$\Longrightarrow$ Uncertainty relations among coordinates emerge

$$\Delta x^\mu \Delta x^\nu \geq \lambda_P^2$$

Learning from quantum mechanics: uncertainty relations can be explained by admitting that coordinate functions be replaced by noncommuting operators

$$[\hat{x}^\mu, \hat{x}^\nu] \neq 0$$

Noncommutative, or Quantum Spacetime

- Spacetime observables (what where smooth functions on classical spacetime) become operators
- States (what where points of classical spacetime, namely "evaluation maps" on the space of classical observables $\omega : f \to f(\omega)$ become "quantum evaluation maps"

Quantum Mechanics as the prototypical NC geometry

- Classical Phase-Space as a differentiable manifold is lost
- Classical observables $\longrightarrow$ Hermitian Operators
- Phase space coordinate functions $q, p \longrightarrow$ noncommuting operators
- The uncertainty principle $\Delta\hat{q}\Delta\hat{p} \geq \frac{\hbar}{2}\mathbb{I}$ implies the existence of a minimal area in phase space
- classical states (points on phase space) $\longrightarrow$ vectors in Hilbert space, or in general, density operators (positive semi-definite, Hermitian operators of trace one)
- Time evolution $\longrightarrow$ dually related pictures: Schrödinger equation for states/ Heisenberg equation for observables

hence

in analogy with the classical case where by Gelfand-Neimark theorem a commutative algebra of continuous functions is enough to reconstruct its underlying topological space and, by adding extra-structure (*spectral triple*) one recovers the whole Riemannian manifold (e.g. classical phase-space)

A noncommutative geometry is a noncommutative algebra with associative product playing the role of a *quantum* space (e.g. quantum phase-space)

The Wigner-Weyl-Moyal approach: The star product and the Weyl-Stratonovich operator

QM can be described in a classical-like setting

- Operators $\longrightarrow$ Symbols (smooth functions on $T^*\mathbb{R}^n$)

$$\hat{A} \longrightarrow f_{\hat{A}}(q,p) = \mathrm{Tr}\,\hat{A}\hat{W}(q,p)$$

  with

$$\hat{W}(q,p) = \int \mathrm{d}\eta \mathrm{d}\xi \, e^{i(\eta\cdot\hat{p}+\xi\cdot\hat{q})} e^{-i(\eta\cdot p+\xi\cdot q)} \qquad (\hbar = 1)$$

  the Weyl-Stratonovich operator or simply quantizer (dequantizer) operator
- states $\hat{\rho} \longrightarrow \mathcal{W}_{\hat{\rho}}(q,p) = \mathrm{Tr}\,\hat{\rho}\hat{W}(q,p)$ the Wigner function
- operator product $\longrightarrow$ star product $\star$ (associative, non-commutative)

$$\hat{A}\hat{B} \longrightarrow f_{\hat{A}} \star f_{\hat{B}}(q,p) = \mathrm{Tr}\,(\hat{A}\hat{B}\hat{W}(q,p))$$

  this yields in particular $q \star p - p \star q = i$
- $(\mathcal{F}(T^*R^n), \star_\hbar)$ prototype NC algebra

Classical gauge theories of fundamental interactions

They describe the interaction between matter and radiation.

Radiation is mathematically described in terms of connection and curvature of principal G-bundles over space-time

$$\begin{tikzcd} P \arrow[d, "\pi"] & G \arrow[l] \\ M & \end{tikzcd}$$

Matter fields as sections $s \in \Gamma(E)$ of vector bundles (E, G, M, π, F) $E \xrightarrow{\pi} M$ associated with the appropriate principal G-bundle

- Relevant example: Electromagnetic interaction $\rightarrow$ $U(1)$-bundle; charged matter $\rightarrow$ sections of 1-dim complex vector bundle carrying a representation of $U(1)$

Standard picture of gauge and matter fields

- $M = (\mathbb{R}^4, \eta)$ space–time
- gauge fields: A_μ, $F_{\mu\nu}$ represent the radiation fields, namely the fields associated with particles which mediate the interactions: electromagnetic, weak, strong, gravitational; they are Lie algebra valued components of forms

  $A = A^a_\mu dx^\mu \tau_a \quad \tau_a \in \mathfrak{g} \qquad \mathfrak{g} = \mathfrak{u}(1), \mathfrak{su}(2), \mathfrak{su}(3)$
  $F = F^a_{\mu\nu} dx^\mu \wedge dx^\nu \tau_a \quad F^a_{\mu\nu} = \partial_\mu A_\nu - \partial A_\mu - i A^b_\mu A^c_\nu f^a_{bc}$

  More formally: $A \in \Omega^1(U) \otimes \mathfrak{g}$ is a Lie algebra valued connection one-form; $F \in \Omega^2(M) \otimes \mathfrak{g}$ is the curvature two-form of A: $F = DA = dA + A \wedge A$
- gauge group: smooth maps from space–time to some unitary Lie group

  $$\widehat{G} = \{g : x \in \mathbb{R}^4 \to g(x) \in G\}$$

  $G = U(1), SU(2), SU(3)$ depending on the interaction; or

  vertical authomorphisms of the G-bundle (automorphisms of P which project to the identity on the base manifold).
- gauge transformations $A' = gAg^{-1} + dgg^{-1}, F' = gFg^{-1}$

- matter fields describing particles (e.g. electrons, protons...) are vector fields on space–time: they are formalised as sections of vector bundles
- what kind of vectors: they carry a representation of the gauge group determined by the interaction they feel; physics says that the representation is the fundamental one (the group characterises the kind of vector bundle)

  electrically charged matter fields are 1-dim complex vector fields (fundamental rep. of the group $U(1)$)
  fields carrying a weak charge are two-dim complex vector fields (fundamental rep. of the group $SU(2)$)
  fields carrying strong charge are three-dim complex vector fields (fundamental rep. of the group $SU(3)$)

  Namely matter fields are organised in multiplets, of dimension depending on the interaction. They can carry more that one representation (e.g. the electron is a 1-dim complex vector field under $U(1)$ but part of a doublet, with its neutrino under $SU(2)$)
- the dynamics is obtained through the linear (Koszul) connection

  $\nabla : X \in \mathfrak{X}(M) \to \nabla_X \in Der(\Gamma(E))$, $\nabla_X = X + \rho[A(X)]$
  with curvature $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = F(X,Y)$ (F the physical field e.g. the electromagnetic tensor of electrodynamics)

To have an idea, the classical action of an electrically charged particle + free action of the radiation field looks like

$$S[\phi, A] = \int \mathrm{d}x^k \, F_{\mu\nu} F^{\mu\nu} + \int \mathrm{d}x^k \, \mathcal{L}(\phi, \nabla_\mu \phi)$$

E.L. equations give Maxwell equations for the electromagnetic field and an evolution equation for the charged particle associated with ϕ

NC theory of gauge and matter fields

[Connes, Dubois-Violette, Grosse, Madore, Wess ....]

The“classical picture" of noncommutative gauge and matter fields is described in terms of

- a noncommutative algebra $(\mathcal{A}, \star)$ representing space-time (it replaces $\mathcal{F}(M)$, hence M)
- a right $\mathcal{A}$-module, $\mathbb{M}$, representing matter fields (it replaces vector bundles)
- a group of unitary automorphisms of $\mathbb{M}$ acting on fields from the left, representing gauge transformations.

The dynamics of fields is described by means of a natural differential calculus based on derivations of the NC algebra;
The gauge connection is the noncommutative analogue of the Koszul connection.
Therefore, the first problem to address is to have a well defined differential calculus, namely, an algebra of $\star$-derivations of $\mathcal{A}$ such that

$$D_a(f \star g) = D_a f \star g + f \star D_a g$$

NC differential calculus

Given the star product of fields in the form

$$f \star g = f \cdot g + \frac{i}{2} \Theta^{ab}(x) \, \partial_a f \partial_b g + \dots$$

ordinary derivations violate the Leibniz rule,

$$\partial_c (f \star g) = (\partial_c f) \star g + f \star (\partial_c g) + \frac{i}{2} \partial_c \Theta^{ab}(x) \, \partial_a f \partial_b g + \dots$$

except for Θ constant in which case star derivations are realised by star commutators

$$D_a f = (\Theta^{-1})_{ab} [x^b, f]_\star = \partial_a f$$

for $\Theta(x) = c_\ell^{jk} x^\ell$ namely Lie algebra type star products, $[x^j, x^k]_\star = c_l^{jk} x^l$ a natural generalisation is

$$D_j f = \kappa [x^j, f]_\star$$

with κ a suitable constant. Alternatively, one can use twisted differential calculus for those NC algebras whose star product is defined in terms of a twist operator.

[Dubois-Violette, Madore, Michor, Masson, Wallet...] It generalises the algebraic description of standard differential calculus to the NC case. In the commutative case vector fields are identified with derivations of $\mathcal{F}(M)$, one-forms and the exterior derivative d are defined by duality

$$df(X) = X(f); \ \alpha = g \cdot df; \ d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$
$$d^2 f(X,Y) = X(df(Y)) - Y(df(X)) - df([X,Y]) = 0$$

Higher forms are constructed analogously.
Thus, to define a differential calculus on a noncommutative algebra, $\mathcal{A}$ we need a Lie algebra $\mathcal{L}$ and a representation of $\mathcal{L}$ in terms of derivations of $\mathcal{A}$. That is, we need $\mathcal{L}$, ρ such that

$$\rho(X)\,(f \star g) = (\rho(X)f) \star g + f \star (\rho(X)g)\,, \quad X \in \mathcal{L}, \quad f,g \in \mathcal{A}$$

Assuming such structures are given, the first step for the construction of a differential calculus is the identification of zero forms with the algebra itself $\Omega^0 = \mathcal{A}$.

Then the exterior derivative is implicitly defined by $df(X) = \rho(X)f$ It automatically verifies the Leibniz rule because $\rho(X)$ are $\star$-derivations

$$d(f \star g)(X) = (\rho(X)f) \star g + f \star (\rho(X)g)$$

moreover $\quad d^2 = 0$
because the $\star$-derivations close a Lie algebra. The second step consists in defining Ω^1 as a left $\mathcal{A}$ module that is

$$gdf(X) = g \star (\rho(X)f)$$

Because of noncommutativity, the wedge product

$$df \wedge_\star dg(X, Y) = df(X) \star dg(Y) - df(Y) \star dg(X)$$

is not anticommutative $df \wedge_\star dg \neq -dg \wedge_\star df$.
In a similar way to Ω^1, Ω^2 is defined as a left $\mathcal{A}$ module, $\omega = f \star dg \wedge_\star dh$
Higher Ω^p are built analogously.
Derivations have to be independent: namely no functions belonging to the center of the algebra exist s.t. $f_\mu X_\mu = 0$ and sufficient, namely if $df(X_\mu) = 0 \ \forall \mu \to f$ is central

The Moyal algebra $\mathcal{A} = \mathbb{R}^k_\theta$

It is the simplest noncommutative space, modelled on the phase space of quantum mechanics:

First, consider the dual description of the classical space $\mathbb{R}^k$ in terms of an appropriate algebra of functions

Then quantize in the Weyl-Wigner-Moyal approach, $f \to \hat{W}(f) := \hat{f}$ and consider the operator symbols $F = \hat{W}^{-1}(\hat{f})$

The algebra of operator symbols is noncommutative, with a star product inherited by the operator product: $(\mathcal{F}(R^k), \star_\theta) =: \mathbb{R}^k_\theta$ is the Moyal algebra

$$f \star g = \hat{W}^{-1}(\hat{f}\hat{g})$$

- The Moyal star product is properly defined for Schwartz functions $\mathcal{S}(\mathbb{R}^k)$ but can be extended so to include polynomials and constants [Varilly-Gracia-Bondia '89]

$$f \star g(x) = \frac{1}{(2\pi)^k} \int \mathrm{d}^k u \mathrm{d}^k v \, f(x - \frac{1}{2}\Theta u) g(x+v) e^{iu\cdot v}$$

Θ is block diagonal, antisymmetric, θ real

$$\Theta = \theta \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & .. \end{pmatrix}$$

$$f \star_\theta g(x) = \exp\left(\tfrac{i}{2}\Theta^{\mu\nu} \tfrac{\partial}{\partial u^\mu}\tfrac{\partial}{\partial v^\mu}\right) f(u) g(v)|_{u=v=x} \quad \Rightarrow \quad [x^\mu, x^\nu]_{\star_\theta} = i\theta^{\mu\nu}$$

Minimal derivation based differential calculus
As a minimal Lie algebra we can choose the Abelian algebra of translations $\{P_\mu\}$ (but we could choose a bigger algebra: the largest algebra of derivations being $\mathfrak{isp}(4,\mathbb{R})$)

$$\rho(P_\mu) := \partial_\mu = -i\theta^{-1}_{\mu\nu}[x^\nu, \cdot]_\star$$

generate the minimal Lie algebra of derivations of $\mathbb{R}^{2n}_\theta$
These are
- inner
- not a left module over $\mathbb{R}^{2n}_\theta$, but only over the center of the algebra because
$$f \star \partial_\mu(g \star h) \neq f \star \partial_\mu g \star h + g \star f \star \partial_\mu h$$
- d, i_{P_μ} defined algebraically,
$$df(P_\mu) = P_\mu(f) = -i\theta^{-1}_{\mu\nu}[x^\nu, f]_\star,$$
$$i_{P_\mu}\omega(P_\nu) = \omega(P_\mu, P_\nu) = f \star (dg(P_\mu) \star dh(P_\nu) - dg(P_\nu) \star dh(P_\mu));$$

Integration
$$\int f \star g = \int g \star f = \int f \cdot g$$

$\Longrightarrow$ the integral is a trace

Linear noncommutativity: the case $\mathbb{R}^3_\lambda$

In order to appreciate the importance of differential calculus consider the case $\Theta = \Theta(x)$. The simplest case is $\Theta^{ij}(x) = c^{ij}_k x^k$, with c^{ij}_k structure constants of some Lie algebra.

An example is the noncommutative space $\mathbb{R}^3_\lambda$ first introduced in [Hammou, Lagraa, Sheikh-Jabbari' 01] as quadratic subalgebra of $\mathbb{R}^4_\theta$. by means of $x^\mu = \frac{1}{2}\bar{z}^a \sigma^\mu_{ab} z^b, \quad \mu = 0,..,3,\ z^1 = x^0 + ix^1, z^2 = x^2 + ix^3$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the generators of $\mathfrak{su}(2)$. The subalgebra generated by x^μ is closed wrt the star product implying

$$[x^i, x^j]_\star = i\lambda \epsilon^{ij}_k x^k$$

and

$$\sum_i {x^i}^2 = {x^0}^2$$

and x^0 star-commutes with x^i. Thus we can alternatively define $\mathbb{R}^3_\lambda$ as the star-commutant of x^0.

The algebra $\mathbb{R}^3_\lambda$

The induced $\star$-product for $\mathbb{R}^3_\lambda$ reads

$$\varphi \star \psi\,(x) = \exp\left[\frac{\lambda}{2}\left(\delta_{ij}x_0 + i\epsilon_{ij}^{k}x_k\right)\frac{\partial}{\partial u_i}\frac{\partial}{\partial v_j}\right]\varphi(u)\psi(v)|_{u=v=x}$$

Analogously to the Moyal algebra one can introduce a matrix basis [V., Wallet '13]:

$$v^j_{m\tilde{m}}(x) = \frac{e^{-2\frac{x_0}{\lambda}}}{\lambda^{2j}}\frac{(x_0+x_3)^{j+m}(x_0-x_3)^{j-\tilde{m}}\,(x_1-ix_2)^{\tilde{m}-m}}{\sqrt{(j+m)!(j-m)!(j+\tilde{m})!(j-\tilde{m})!}} \quad j\in\frac{\mathbb{N}}{2},\; m,\tilde{m}\in(-j,j)$$

$\Longrightarrow$

$$v^j_{m\tilde{m}} \star v^{\tilde{j}}_{n\tilde{n}}(x) = \delta^{j\tilde{j}}\delta_{m\tilde{m}}$$

Then, the star product in $\mathbb{R}^3_\lambda$ becomes a block-diagonal infinite-matrix product and the integral becomes a trace.
This is important for physical applications.

Derivations of the algebra $\mathbb{R}^3_\lambda$

The identification of the algebra $\mathbb{R}^3_\lambda$ as a subalgebra of $\mathbb{R}^4_\theta$ has a geometric counterpart in the commutative setting, where the Kustaanheimo-Stiefel (KS) map can be used:

- $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are given the structure of trivial bundles over spheres, $\mathbb{R}^3 - \{0\} \simeq S^2 \times \mathbb{R}^+$, $\mathbb{R}^4 - \{0\} \simeq S^3 \times \mathbb{R}^+$;
- then use the Hopf fibration $\pi_H : S^3 \to S^2$ and extend the Hopf map to $\mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$.
- Projectable vector fields are defined by the condition $[D_a, Y_0] = 0, a = 1, 4$, with Y_0 the generator of the fibre $U(1)$.
- They correspond to the three rotation generators Y_i and the dilation D.

$$\pi_{KS*}(Y_i) = X_i = \epsilon_{ijk} x_j \frac{\partial}{\partial x_k}\,, \quad \pi_{KS*}(D) = x_i \frac{\partial}{\partial x_i}$$

The three rotations are not independent since $x_i \cdot X_i = 0$, nor sufficient because $X_i(f(x^0)) = 0$, but x^0 not a constant in $\mathbb{R}^3$. Problem solved by adding the dilation D.

When passing to the noncommutative case the three rotations are still derivations of the algebra $\mathbb{R}^3_\lambda$ and may be given the form of inner derivations

$$X_i(\varphi) = -\frac{i}{\lambda}[x_i, \varphi]_\star \,, \quad i = 1, .., 3$$

- they satisfy the Leibniz rule
- they are now independent (even though $x_i \star X_i(\varphi) + X_i(\varphi) \star x_i = 0$, derivations are not a module over the algebra in the NC case)
- and sufficient (constant functions under Y_i are only those in the center of the algebra)

The dilation is not a star-derivation as it does not satisfy the Leibniz rule.

Noncommutative gauge theory on Moyal space

To make sense of noncommutative gauge and matter fields we need

- ✓ a noncommutative algebra $(\mathcal{A}, \star)$ representing space-time (it replaces $\mathcal{F}(M)$)
- ✓ A differential calculus based on derivations of the NC algebra which allows to introduce the dynamics;
- a NC analogue of matter fields, compatible with $\star$ multiplication by functions, which replaces the notion of vector bundles
- a group of unitary automorphisms acting on fields from the left, representing gauge transformations;
- a NC analogue of gauge connection

For QED the gauge group is $\widehat{U(1)}$, implying that charged matter fields are 1-dim complex vector fields (sections of 1-d complex vector bundle), namely a right module over $\mathcal{F}(\mathbb{R}^4)$

$\Longrightarrow$ The NC generalization is

- a 1-dim complex right module (one generator) over $\mathbb{R}^{2n}_\theta$

$$\mathcal{H} = \mathbb{C} \otimes \mathbb{R}^{2n}_\theta$$

with Hermitian structure $h : h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = \psi_1^\dagger \star \psi_2$

Non-Abelian gauge theories

For non-Abelian gauge theories (gauge group $\widehat{SU(N)}$) charged matter fields are typically complex vector fields in the fundamental representation of the group (–> sections of N-dim complex vector bundles)

$\Longrightarrow$ The NC generalization is

- a N-dim complex right module (N generators) over $\mathbb{R}^{2n}_\theta$

$$\mathcal{H} = \mathbb{C}^N \otimes \mathbb{R}^{2n}_\theta$$

- Gauge transformations are defined as automorphisms of $\mathcal{H}$ compatible both with the structure of right $\mathbb{R}^{2n}_\theta$-module

$$g(\psi f) = g(\psi) f$$

and with the Hermitian structure $h : \mathcal{H} \times \mathcal{H} \to \mathbb{R}^{2n}_\theta$

$$h(g\psi_1), g(\psi_2)) = h(\psi_1, \psi_2) \quad \forall \psi_1, \psi_2 \in \mathcal{H}$$

- A linear connection is a linear map $\nabla : \mathrm{Der}(\mathbb{R}^{2n}_\theta) \times \mathcal{H} \to \mathcal{H}$ satisfying
  - $\nabla_X(\psi f) = \psi X(f) + \nabla_X(\psi) f, \nabla_{cX}(\psi) = c\nabla_X(\psi)$ $\quad$ c in the center
  - $\nabla_{X+Y}(\psi) = \nabla_X(\psi) + \nabla_Y(\psi)$
  - Hermiticity:
    $X(h(\psi_1, \psi_2)) = h(\nabla_X(\psi_1), \psi_2) + h(\psi_1, \nabla_X(\psi_2)), \forall \psi_1, \psi_2 \in \mathcal{H}$

- Curvature is the linear map $\mathrm{F}(X, Y) : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathrm{F}(X, Y)\psi = i\,([\nabla_X, \nabla_Y]\psi - \nabla_{[X,Y]})\psi$$

Noncommutative Electrodynamics on R^{2n}_θ

In this case $\mathcal{H}$ has only one generator, e $\longrightarrow$ $\boldsymbol{\psi} = \mathrm{e}\psi, \psi \in R^{2n}_\theta$

- The connection is completely specified by its value on the module generator:

$\nabla_X(\boldsymbol{\psi}) = \nabla_X(\mathrm{e})\psi + \mathrm{e}X(\psi)$, with $\nabla_X(\mathrm{e})^\dagger = -\nabla_X(\mathrm{e})$.

$\Longrightarrow$ The 1-form connection A:

- $\mathrm{A} : X \to \mathrm{A}(X) := i\nabla_X(\mathrm{e}), \;\; \forall X \in \mathrm{Der}(\mathbb{R}^{2n}_\theta)$
- $\nabla_\mu(\mathrm{e}) =: -i\mathrm{A}(\partial_\mu) = -i\mathrm{e}A_\mu$
- so that

$$\nabla_\mu \boldsymbol{\psi} := \nabla_\mu(\mathrm{e}\psi) = \mathrm{e}(\partial_\mu\psi - iA_\mu \star \psi)$$

- Gauge transformations can be identified with the unitaries $\mathcal{U}(\mathbb{R}^{2n}_\theta)$

Indeed

$$g(\boldsymbol{\psi}) = g(\mathrm{e}\psi) = g(\mathrm{e}) \star \psi = \mathrm{e}\, f_g \star \psi$$
$$h(g(\boldsymbol{\psi}_1), g(\boldsymbol{\psi}_2)) = h(\mathrm{e},\mathrm{e})\overline{(f_g \star \psi_1)} \star f_g \star \psi_2 = h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \longrightarrow$$
$$\bar{f}_g \star f_g = 1$$
$$\Longrightarrow f_g \in \mathcal{U}(\mathbb{R}^{2n}_\theta)$$

Properties of the gauge connection

- gauge covariance: $(\nabla^A_\mu)^g(\psi) := g(\nabla^A_\mu(g^{-1}\psi)) = \nabla^{A^g}_\mu(\psi)$
  with
  $A^g_\mu = f_g \star A_\mu \star f_{g^{-1}} + i f_g \star \partial_\mu f_{g^{-1}}$
- Curvature:
  $\mathsf{F}_{\mu\nu} = ([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[\partial_\mu,\partial_\nu]}) = \mathrm{e}(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star)$
  $\mathsf{F}^g_{\mu\nu} = ([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[\partial_\mu,\partial_\nu]}) \overset{\text{check}}{=} \mathrm{e}(f_g \star F_{\mu\nu} \star f_{g^{-1}})$

Implying
$F^g_{\mu\nu} \star F^g{}_{\mu\nu} = f_g \star F_{\mu\nu} \star F_{\mu\nu} \star f_{g^{-1}}$

The action functional on R_θ^{2n}

A natural candidate is

$$S = \int d^{2n}x \;\; F_{\mu\nu} \star F^{\mu\nu}$$

Symmetries

- because of cyclicity of the product it is gauge invariant
- it is invariant under standard observer Poincaré transformations
- but path integral quantization yields new pathologies w.r.t. the commutative case: UV/IR mixing, Gribov ambiguity

Space-time symmetries

Moyal product has been shown to be covariant under observer (passive) transformations belonging to the Weyl group (*undeformed* Poincaré + dilations; -more generally under linear affine transformations-) [A. Gaumé '06, GraciaBondia-R.Ruiz-Lizzi-V.'06]

$$[\Omega \cdot f] \star_{\Omega\cdot\Theta} [\Omega \cdot g] = \Omega \cdot (f \star_\Theta g), \quad \Omega = (L, a)$$

$$[\Omega \cdot f](x) = f(L^{-1}(x - a)), \quad \Omega \cdot \Theta = L \Theta L^t$$

Infinitesimal generators:

- They are the standard ones $G = \epsilon^{\alpha}_{\beta} x^{\beta}\partial_{\alpha} + a^{\beta}\partial_{\beta}$
- not derivations of the star product (precisely because the Lie derivative of Θ has to be taken into account)
- However: since the product depends on Θ even if starting functions don't, it is convenient to consider a (x, Θ)-space on which

$$\Omega \cdot (x, \Theta) = (Lx + a, L\Theta L^{t}) \Longrightarrow$$

the infinitesimal generators in (x, Θ)-space are

$$P^{\Theta}_{\mu} = -\partial_{\mu}, \quad D^{\Theta} = -x\cdot\partial - \theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}$$
$$M^{\Theta}_{\mu\nu} = x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} + \theta^{\rho}_{\mu}\frac{\partial}{\partial\theta^{\rho\nu}} - \theta^{\rho}_{\nu}\frac{\partial}{\partial\theta^{\rho\mu}}$$

They close the standard Weyl algebra and are derivations of the star product

$$G^{\theta}(f\star g) = G^{\theta}f\star g + f\star G^{\theta}g$$

Twisted symmetries

Moyal product is not covariant under Poincaré *particle* (active) transformations, where the background field Θ does not change.
But it is covariant under θ-Poincaré particle transformations: the universal enveloping algebra of the Lie algebra $\mathfrak{p}$, with twisted coproduct (Hopf algebra $U_{\mathcal{F}}(\mathfrak{p})$).

A Hopf algebra $H(\mu, \eta, \Delta, \epsilon, S)$ (examples: $U(\mathfrak{g})$, $C^\infty(G)$), is a structure composed by

- a unital associative algebra (H, μ, η)
- a counital coassociative coalgebra (H, Δ, ϵ)

i.e. a vector space H over $\mathbb{C}$ with the following

- μ :H$\otimes H \to H$ the multiplication map
- $\eta : \mathbb{C} \to H$ the unit map
- $\Delta : H \to H \otimes H$ the coproduct
- $\epsilon : H \to \mathbb{C}$ the counit map
- $S : H \to H$ the antipode (generalises the inverse of an element)

with a series of compatibility conditions

Relevant examples for us

- $U(\mathfrak{g})$: $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$, $S(x) = -x$
- $C^\infty(G)$: $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g^{-1}$

The *twist operator* is an invertible element $\mathcal{F}$ in $H \otimes H$ that satisfies the conditions

$$(1 \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes 1)(\Delta \otimes \mathrm{id})\mathcal{F} \qquad (\epsilon \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \epsilon)\mathcal{F} = 1 \otimes 1$$

$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}$, with h in H, defines a new coproduct in H

The algebra underlying H endowed with $\Delta_{\mathcal{F}}$ is the Hopf algebra $H_{\mathcal{F}}$ (twisted Hopf algebra)

If H has a representation in an associative algebra $\mathcal{A}$ (here $F(\mathbb{R}^4)$) with product m:

$$m(a \otimes b) = ab$$
$$h \cdot (ab) = h \cdot m\,(a \otimes b) = m\big(\Delta(h) \cdot (a \otimes b)\big)\,, \quad h \in H$$

the twisting of Δ introduces in $\mathcal{A}$ a twisted product $m_{\mathcal{F}}$ defined by

$$m_{\mathcal{F}}(a \otimes b) = m\big(\mathcal{F}^{-1} \cdot (a \otimes b)\big)$$

which is associative and twist-covariant:

$$\begin{aligned} h \cdot m_{\mathcal{F}}(a \otimes b) &= h \cdot m\big(\mathcal{F}^{-1} \cdot (a \otimes b)\big) = m\big(\Delta(h)\mathcal{F}^{-1} \cdot (a \otimes b)\big) \\ &= m\big(\mathcal{F}^{-1}\Delta_{\mathcal{F}}(h) \cdot (a \otimes b)\big) = m_{\mathcal{F}}\big(\Delta_{\mathcal{F}}(h) \cdot (a \otimes b)\big) \quad ** \end{aligned}$$

$\Longrightarrow$ A $\star$-product defined in terms of a twist is *always* twist-covariant, by definition
$\Longrightarrow$ *An action functional invariant under some space-time transformations always yields a twisted action invariant wrt the corresponding twisted transformations; these should be understood as particle (active) transformations*

The twist operator for the Moyal algebra $\mathbb{R}^4_\Theta$

Consider the Lie algebra of diffeomorphisms, $\mathfrak{D}(\mathbb{R}^4)$, whose generators are vector fields with polynomial coefficients on $\mathbb{R}^4$

- As Hopf algebra H take the enveloping algebra $U(\mathfrak{D})$:
  Δ is first defined for $h \in \mathfrak{D}$ by $\Delta(h) = 1 \otimes h + h \otimes 1$, and then multiplicatively extended to all of $U(\mathfrak{D})$ by $\Delta(hh') = \Delta(h)\Delta(h')$;
- for the algebra $\mathcal{A}$ carrying a representation of $U(\mathfrak{D})$, take the algebra of functions on spacetime with the ordinary multiplication $m(f \otimes g) = fg$;
- for $\mathcal{F}$, take $\mathcal{F}_\Theta = \exp(-\frac{i}{2}\,\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu)$. This is clearly in $U(\mathfrak{D}) \otimes U(\mathfrak{D})$, has an inverse
  $$\mathcal{F}_\Theta^{-1} = \exp(\tfrac{i}{2}\,\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu)$$
  and satisfies the cocycle condition

The Moyal product is then recovered as the twisted product

$$m_\Theta(f \otimes g) = m\big(\mathcal{F}_\Theta{}^{-1} \cdot (f \otimes g)\big) = f \star_\Theta g$$

The action of a generator h on the Moyal product is determined by $\Delta_\Theta(h) = \mathcal{F}_\Theta \Delta(h) \mathcal{F}_\Theta{}^{-1}$ and conversely.

For the generators of translations, Lorentz transformations and dilations the following expressions were obtained [Kulish, Matlock]

$$
\begin{aligned}
\Delta_\Theta(P_\mu) &= P_\mu \otimes 1 + 1 \otimes P_\mu \\
\Delta_\Theta(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\
&\quad + \tfrac{i}{2}\,\theta^{\alpha\beta}\big[(g_{\mu\alpha}P_\nu - g_{\nu\alpha}P_\mu) \otimes P_\beta + P_\alpha \otimes (g_{\mu\beta}P_\nu - g_{\nu\beta}P_\mu)\big] \\
\Delta_\Theta(D) &= D \otimes 1 + 1 \otimes D - i\,\theta^{\mu\nu} P_\mu \otimes P_\nu
\end{aligned}
$$

From these formulas it was concluded that Poincaré invariance can be maintained in noncommutative field theory although twisted.

But this is not specific of Poincaré invariance

Note that Eq. $**$ places *no restriction* on the generator h except that of being an infinitesimal diffeomorphism

Twisted differential calculus

The principle adopted is that *every bilinear map* should be consistently deformed [Aschieri-V.-Lizzi '08] by composing it with the twist

$$\mu : A \times B \to C \Longrightarrow \mu_\star = \mu \circ \mathcal{F}^{-1}$$

- the wedge product of two forms of arbitrary degree, ω_1 and ω_2, is deformed into the $\star$-wedge product:

$$(\omega_1 \wedge_\star \omega_2)(x) = \mathcal{F}^{-1}(y,z)\omega_1(y) \wedge \omega_2(z)\Big|_{x=y=z}$$

- For Moyal twist the usual (commutative) exterior derivative satisfies:

$$\begin{aligned} \mathrm{d}(f \star g) &= \mathrm{d}f \star g + f \star \mathrm{d}g, \\ \mathrm{d}^2 &= 0 \end{aligned}$$

fulfilled because it commutes with Lie derivatives that enter in the definition of the $\star$-product

- If the twist is Abelian (like Moyal twsit), the cyclicity of the integral holds

$$\int \omega_1 \wedge_\star \cdots \wedge_\star \omega_p = (-1)^{d_1 \cdot d_2 \cdots d_p} \int \omega_p \wedge_\star \omega_1 \wedge_\star \cdots \wedge_\star \omega_{p-1},$$

  with $d_1 + d_2 + \cdots + d_p = 4$. It can be shown that the twist fulfils an even stronger requirement. Namely, one can check that the $\star$-product of functions is indeed closed

$$\int \mathrm{d}^4 x\, f \star g = \int \mathrm{d}^4 x\, g \star f = \int \mathrm{d}^4 x\, f \cdot g$$

The last property in general does not hold for coordinate dependent $\star$-products, as for example κ-Minkowski $\star$products or $\mathfrak{su}(2)$ ones

NC Electrodynamics on $\mathbb{R}^3_\lambda$

- The star product of R^3_λ

$$\varphi \star \psi\,(x) = \exp\left[\frac{\lambda}{2}\left(\delta_{ij}x_0 + i\epsilon_{ij}^k x_k\right)\frac{\partial}{\partial u_i}\frac{\partial}{\partial v_j}\right]\varphi(u)\psi(v)|_{u=v=x}$$

- The matrix basis
  - An orthogonal matrix basis with $j \in \frac{\mathbb{N}}{2}, -j \leq m, \tilde{m} \leq j$

$$v^j_{m\tilde{m}}(x) = \frac{e^{-2\frac{x_0}{\lambda}}}{\lambda^{2j}}\frac{(x_0+x_3)^{j+m}(x_0-x_3)^{j-\tilde{m}}\,(x_1-ix_2)^{\tilde{m}-m}}{\sqrt{(j+m)!(j-m)!(j+\tilde{m})!(j-\tilde{m})!}}$$

  - $v^j_{m\tilde{m}} \star v^{\tilde{j}}_{n\tilde{n}} = \delta^{j\tilde{j}}\delta_{\tilde{m}n}v^j_{m\tilde{n}}$
  - $\int v^j_{m\tilde{m}} = C\delta_{m,\tilde{m}}$ with $\int \longrightarrow \mathrm{Tr}$
- The derivation based differential calculus and the gauge connection
  - derivations are inner $D_i := \frac{i}{\lambda^2}[x^i, \cdot]_\star, \quad i = 1, \dots, 3$
  - A gauge connection is defined as previously on $\mathcal{H} = \mathbb{C} \otimes \mathbb{R}^3_\lambda$

$$\nabla_{D_i}\boldsymbol{\varphi} = \nabla_{D_i}(\mathrm{e}) \star \varphi + \mathrm{e}D_i\varphi \longrightarrow \mathrm{A}_i = i\nabla_{D_i}(\mathrm{e})$$

Models of QED on $\mathbb{R}^3_\lambda$ have been studied in perturbative expansion (one loop)

Twist approach: Angular noncommutativity (λ-Minkowski)

Space–time noncommutativity is given by

$$[\hat{x}^3, \hat{x}^1] = -i\lambda\hat{x}^2, \quad [\hat{x}^3, \hat{x}^2] = i\lambda x^1, \quad [\hat{x}^1, \hat{x}^2] = [\hat{x}^0, \hat{x}^i] = 0$$

Properties

- there exists a star product reproducing coordinates non-commutativity, deriving from a twist operator $\mathcal{F} \in \mathfrak{p} \otimes \mathfrak{p}$
- although the commutation relations violate Poincaré symmetry (active and passive), the symmetry can be twisted —> a twisted λ-Poincaré Hopf algebra can be defined
- $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F} &= \exp\left\{-\frac{i\lambda}{2}\left(\partial_{x^3} \otimes \left(x^2\partial_{x^1} - x^1\partial_{x^2}\right) - \partial_{x^3} \otimes \left(x^2\partial_{x^1} - x^1\partial_{x^2}\right)\right)\right\} \\ &= \exp\left\{\frac{i\theta}{2}\left(\partial_{x^3} \otimes \partial_{\varphi} - \partial_{x^3} \otimes \partial_{\varphi}\right)\right\} \end{aligned}$$

with $x^1 = \rho\cos\varphi$, $x^2 = \rho\sin\varphi$

The star product

Since the vector fields ∂_φ and ∂_3 commute, the twist is admissible, because it satisfies the cocycle condition $\to$ the associated $\star$ product is associative;

$$(f \star g)(x) = m \circ \mathcal{F}^{-1}(f \otimes g)(x) = fg - \frac{i\lambda}{2}(\partial_\varphi f \partial_3 g - \partial_3 f \partial_\varphi g) + O(\lambda^2).$$

Notice that the role of x_3 and x_0 can be exchanged. Algebraically not a problem, but physically it makes a big difference, if x_0 is time

- The Abelian twist $\mathcal{F}$ is a special example of a more general twist introduced in [Lukierski& coll. '94]
- The NC differential geometry induced by $\mathcal{F}$ was constructed in [Konjik-Dimitriejivic-Samsarov '17]

In cylindrical coordinates

$$[x^3, \rho]_\star = 0\,, \quad [x^3, e^{i\varphi}]_\star = -\lambda e^{i\varphi}\,, \quad [x^3, f(x^0, x^3, \rho, \varphi)]_\star = i\lambda \partial_\varphi f$$

For field theory it is useful to calculate the $\star$-product of two plane waves. We have

$$e^{-ip\cdot x} \star e^{-iq\cdot x} = e^{-i(p+_\star q)\cdot x},$$

where the $\star$-sum of the 4-momenta is defined as follows:

$$p +_\star q = R(q_3)p + R(-p_3)q,$$

and R is the following matrix:

$$R(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\lambda t}{2}\right) & \sin\left(\frac{\lambda t}{2}\right) & 0 \\ 0 & -\sin\left(\frac{\lambda t}{2}\right) & \cos\left(\frac{\lambda t}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

it corresponds to a rotation matrix in the $(p_1 p_2)$ plane; the angle of rotation is proportional to the noncommutativity parameter, and to the momenta involved; it reduces to the identity in the commutative limit $\lambda \longrightarrow 0$ as well as in the low momentum limit.

- it can be checked that the $\star$-sum is noncommutative, but associative and satisfies $p +_\star (-p) = 0$ for an arbitrary 4-vector p;
- generalizing to the product of three plane waves,

$$e^{-ip\cdot x} \star e^{-iq\cdot x} \star e^{-ir\cdot x} = e^{-i(p+_\star q+_\star r)\cdot x},$$

with

$$p +_\star q +_\star r = R(r_3 + q_3)p + R(-p_3 + r_3)q + R(-p_3 - q_3)r$$

by induction:

$$p^{(1)} +_\star \ldots +_\star p^{(N)} = \sum_{j=1}^{N} R\left(-\sum_{1\le k<j} p_3^{(k)} + \sum_{j<k\le N} p_3^{(k)}\right) p^{(j)}$$

It can be shown that the $\star$-sum can be related to the twisted coproduct of momenta P_μ in the twisted Poincaré Hopf algebra, with previous angular twist

The twisted Poincaré algebra [Dimitriejivic-Konjik-Samsarov '17]

Poincaré generators:

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ M_{\mu\nu} &= i(\eta_{\mu\lambda}x^\lambda\partial_\nu - \eta_{\nu\lambda}x^\lambda\partial_\mu) \end{aligned}$$

with $\eta_{\mu\nu} = (+,-,-,-)$ and comm. relations

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu),$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho})$$

twisted coproduct of momenta

$$\begin{aligned} \Delta^{\mathcal{F}} P_0 &= P_0 \otimes 1 + 1 \otimes P_0 \\ \Delta^{\mathcal{F}} P_3 &= P_3 \otimes 1 + 1 \otimes P_3 \\ \Delta^{\mathcal{F}} P_1 &= P_1 \otimes \cos\left(\frac{\theta}{2}P_3\right) + \cos\left(\frac{\theta}{2}P_3\right) \otimes P_1 + P_2 \otimes \sin\left(\frac{\theta}{2}P_3\right) - \sin\left(\frac{\theta}{2}P_3\right) \otimes P_2 \\ \Delta^{\mathcal{F}} P_2 &= P_2 \otimes \cos\left(\frac{\theta}{2}P_3\right) + \cos\left(\frac{\theta}{2}P_3\right) \otimes P_2 - P_1 \otimes \sin\left(\frac{\theta}{2}P_3\right) + \sin\left(\frac{\theta}{2}P_3\right) \otimes P_1 \end{aligned}$$

twisted coproduct of Lorentz generators:

$$
\begin{aligned}
\Delta^{\mathcal{F}} M_{31} = {} & M_{31} \otimes \cos\left(\frac{\theta}{2} P_3\right) + \cos\left(\frac{\theta}{2} P_3\right) \otimes M_{31} + M_{32} \otimes \sin\left(\frac{\theta}{2} P_3\right) - \sin\left(\frac{\theta}{2} P_3\right) \otimes M_{32} \\
& - P_1 \otimes \frac{\theta}{2} M_{12} \cos\left(\frac{\theta}{2} P_3\right) + \frac{\theta}{2} M_{12} \cos\left(\frac{\theta}{2} P_3\right) \otimes P_1 \\
& - P_2 \otimes \frac{\theta}{2} M_{12} \sin\left(\frac{\theta}{2} P_3\right) - \frac{\theta}{2} M_{12} \sin\left(\frac{\theta}{2} P_3\right) \otimes P_2 \\
\Delta^{\mathcal{F}} M_{32} = {} & M_{32} \otimes \cos\left(\frac{\theta}{2} P_3\right) + \cos\left(\frac{\theta}{2} P_3\right) \otimes M_{32} - M_{31} \otimes \sin\left(\frac{\theta}{2} P_3\right) + \sin\left(\frac{\theta}{2} P_3\right) \otimes M_{31} \\
& - P_2 \otimes \frac{\theta}{2} M_{12} \cos\left(\frac{\theta}{2} P_3\right) + \frac{\theta}{2} M_{12} \cos\left(\frac{\theta}{2} P_3\right) \otimes P_2 \\
& + P_1 \otimes \frac{\theta}{2} M_{12} \sin\left(\frac{\theta}{2} P_3\right) + \frac{\theta}{2} M_{12} \sin\left(\frac{\theta}{2} P_3\right) \otimes P_1 \\
\Delta^{\mathcal{F}} M_{30} = {} & M_{30} \otimes 1 + 1 \otimes M_{30} - \frac{\theta}{2} P_0 \otimes M_{12} + \frac{\theta}{2} M_{12} \otimes P_0 \\
\Delta^{\mathcal{F}} M_{12} = {} & M_{12} \otimes 1 + 1 \otimes M_{12} \\
\Delta^{\mathcal{F}} M_{10} = {} & M_{10} \otimes \cos\left(\frac{\theta}{2} P_3\right) + \cos\left(\frac{\theta}{2} P_3\right) \otimes M_{10} + M_{20} \otimes \sin\left(\frac{\theta}{2} P_3\right) - \sin\left(\frac{\theta}{2} P_3\right) \otimes M_{20} \\
\Delta^{\mathcal{F}} M_{20} = {} & M_{20} \otimes \cos\left(\frac{\theta}{2} P_3\right) + \cos\left(\frac{\theta}{2} P_3\right) \otimes M_{20} - M_{10} \otimes \sin\left(\frac{\theta}{2} P_3\right) + \sin\left(\frac{\theta}{2} P_3\right) \otimes M_{10}
\end{aligned}
$$

The coproducts of momenta P_0 and P_3 and of M_{12}, the generator of the rotation in the x^1x^2 plane, remain undeformed (primitive); all other coproducts are deformed

The $\lambda\phi^{\star 4}$ model [Dimitrijevic-Konjik-Kurkov-Lizzi-V. '18]

The scalar field theory theory on λ-Minkowski described by

$$S = \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2}\partial_\mu\phi(x) \star \partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)\star\phi(x) - \frac{\lambda}{4!}\phi(x)^{\star 4} \right)$$

- because of the closure of the $\star$-product, it is possible to replace the $\star$-product in all quadratic terms by the usual (pointwise) one;
- as a consequence the free propagators are the same as in the commutative theory, but not the vertex;

Deformed Conservation of Momentum

Expanding the field $\phi(x)$ in its Fourier modes

$$\phi(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} d^4p\, e^{-ipx} \widetilde{\phi}(p)$$

one arrives at the following expression for the classical action in momentum space

$$\begin{aligned} S \;=\; & \int_{\mathbb{R}^4 \times \mathbb{R}^4} dp\, dq\, \frac{1}{2} \left(-p_\mu q^\mu \widetilde{\phi}(p)\widetilde{\phi}(q) - m^2 \widetilde{\phi}(p)\widetilde{\phi}(q) \right) \delta^{(4)}\left(p +_\star q\right) \\ - \; & \frac{1}{(2\pi)^4} \frac{\lambda}{4!} \int_{(\mathbb{R}^4)^{\times 4}} dp\, dq\, dr\, ds\; \widetilde{\phi}(p)\widetilde{\phi}(q)\widetilde{\phi}(r)\widetilde{\phi}(s) \delta^{(4)}\left(p +_\star q +_\star r +_\star s\right) \end{aligned}$$

- *the only* difference wrt to the commutative case is the presence of the $\star$-sum in the delta functions;
- these δ functions encode the conservation of momentum in the corresponding vertices
- therefore the main difference is the twisted conservation of momentum;
- by computing the one-loop corrections to the propagator (planar and non planar diagram) we find UV/IR mixing

Summary

- We have reviewed the mathematical framework to describe NC gauge (and field) theory within two main approaches:
  - the derivation based differential calculus
  - the twist approach
- in the NC setting the definition of symmetries gets modified; we have reviewed full covariance vs twist-covariance in relation with observer-dependent and particle-dependent symmetries;
- a powerful approach is represented by the use of matrix bases: we have said very little about them; the very first full proof of renormalizability to all orders of a NC field theory has been done in the matrix basis [Grosse-Wulkenhaar 'xx];
- we have not touched upon Wick rotation: it is a delicate issue for those models with time noncommutativity

Open problems and Perspectives

- The quantum theory:

  The picture I have described is classical. The theory has to be quantized. Problems with renormalizability

- The commutative limit
  - Most of the proposals for NC generalizations of gauge theories do not have the correct commutative limit
  - There are approaches based on L_∞ algebras
  - Others based on symplectic realizations and groupoids ....