Universität Münster

Noncommutative geometry meets topological recursion

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CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

MM Mathematics Münster Cluster of Excellence

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- enumerative geometry and combinatorics,
- quantum gravity in two dimensions,
- complex algebraic geometry,
- quantum fields on noncommutative geometry,

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They are examples for a universal structure called topological recursion.

We try to give an idea of recently established connections between topological recursion and the two above fields related to noncommutative geometry:
(1) free probability and (2) QFT on NCG.

Topological recursion

[Eynard, Orantin 07] noticed that the non-linearity of many matrix models can be disentangled into initial data called the spectral curve and a universal recursion for meromorphic functions $\mathcal{W}_n^{(g)}$ (or promoted to meromorphic differentials $\omega_n^{(g)} = \mathcal{W}_n^{(g)} \prod dx$).

Spectral curve $(x, y : \Sigma \to \mathbb{P}^1, B)$

- Complex curve/Riemann surface Σ and two ramified coverings $x, y : \Sigma \to \mathbb{P}^1$. Polynomial equation $P(x, y) = 0$.
- Bergman kernel B: symmetric bidifferential on $\Sigma \times \Sigma$, with double pole on diagonal, no other pole, normalised.

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Soon later many important examples other than matrix models were identified:

- Weil-Peterssen volumes of moduli spaces of bordered hyperbolic surfaces [Mirzakhani 07].
- ELSV formula, expresses simple Hurwitz numbers as integral of ψ- and λ-classes over $\overline{\mathcal{M}}_{g,n}$ [Bouchard, Mariño 07; Eynard, Mulase, Safnuk 09].
- Semisimple cohomological field theories [Dunin-Barkowski, Orantin, Shadrin, Spitz 14].

Free probability

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Introduced by Voiculescu in the 80s to attack the free group factor problem.

Definition

A noncommutative probability space is a unital algebra $\mathcal{A}$ together with a unital functional $\varphi : \mathcal{A} \to \mathbb{C}$. A family of subalgebras $\mathcal{A}_i \subseteq \mathcal{A}$ is called *free* if $\varphi(a_1 \cdots a_n) = 0$ whenever $\varphi(a_k) = 0$ ('centered') and neighbouring a_k are from different subalgebras $\mathcal{A}_{i_k}$.

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Free central limit theorem [Voiculescu 85]

Let $(a_i)_{i\in\mathbb{N}}$ be a family of freely independent random variables in nc prob. space $(\mathcal{A}, \varphi)$ which are (1) identically distributed $\varphi(a_i^r) = \varphi(a_j^r)$, (2) centered $\varphi(a_i) = 0$, and (3) normalised $\varphi(a_i^2) = 1$. Let $S_n := \frac{1}{\sqrt{n}}(a_1 + \ldots + a_n)$. Then $(S_n)_{n\in\mathbb{Z}_{>0}}$ converges to semicircular variable:

$$\lim_{n\to\infty} \varphi(S_n^k) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} dt\, t^k \sqrt{4-t^2} = \begin{cases} 0 & \text{for } k \text{ odd,} \\ C_m = \frac{1}{m+1}\binom{2m}{m} & \text{for } k = 2m \text{ even.} \end{cases}$$

Eigenvalue distribution of large random GUE matrices obeys same semicircular law [Wigner 55].

Free moments and free cumulants

Definition [Speicher 92]

The free cumulants $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ in nc prob. space $(\mathcal{A}, \varphi)$ are inductively defined by

$$\varphi(a_1 \cdots a_n) =: \sum_{\substack{\pi \in \text{non-crossing} \\ \text{partitions of } n}} \prod_{\text{blocks } V \in \pi} \underbrace{\kappa_l(a_{i_1}, \ldots, a_{i_l})}_{\text{if } V = (i_1, \ldots, i_l)}$$

Freeness is equivalent to vanishing of mixed cumulants: $\kappa_n(a_1, \ldots, a_n) = 0$, $n \geq 2$, whenever some $\{a_i, a_j\}$ from different subalgebras.

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Theorem

For $a \in \mathcal{A}$, the generating series

$$M_a(z) := 1 + \sum_{n=1}^{\infty} \varphi(a^n) z^n \;, \qquad C_a(z) := 1 + \sum_{n=1}^{\infty} \kappa_n(a, \dots, a) z^n$$

satisfy the functional relation $C_a(zM_a(z)) = M_a(z)$

(equivalent to a formula for Voiculescu's R-transform)

Higher-order freeness

[Collins, Mingo, Śniady, Speicher 06/07] introduced freeness of higher order and corresponding moments and cumulants, collected to

$$M_n(X_1, \dots, X_n) := \sum_{k_1,\dots,k_n=1}^{\infty} \varphi_n(a^{k_1}, \dots, a^{k_n}) \prod_{i=1}^{n} X_i^{k_i}, \quad C_n(y_1, \dots, y_n) := \sum_{k_1,\dots,k_n=1}^{\infty} \kappa_{k_1,\dots,k_n}(a, \dots, a) \prod_{i=1}^{n} y_i^{k_i}$$

They were able to identify the functional relation for $n = 2$:

$$M_2(X_1, X_2) + \frac{X_1 X_2}{(X_1 - X_2)^2} = \frac{d \log y_1}{d \log X_1} \frac{d \log y_2}{d \log X_2} \Big(C_2(y_1, y_2) + \frac{y_1 y_2}{(y_1 y_2)^2} \Big) \qquad (*)$$

where $y_i = X_i M_1(X_1)$ and $X_i = y_i / C_1(y_1)$. The generalisation to $n \geq 3$ remained open.

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- [Borot, Garcia-Failde 17] proved that (*) also relates generating series for ordinary and fully simple maps, themselves governed by TR. Is free probability linked to TR?
- [Borot, Charbonnier, Garcia-Failde, Leid, Shadrin 21] identified several facets of a master relation, one facet leading to functional relations for (g,n)-free moments and cumulants.

Master relation and functional relations

- Free probablity is rooted in permutations, partitions and partitioned permutations.
- Partioned permutation come with a product, which induces a convolution product $\star$ on functions of partioned permutations.
- An important such function is the zeta function, $\zeta(\mathcal{A}, \alpha) = 1$ iff the partition $\mathcal{A}$ of $\{1, ..., d\}$ equals the support of the cycles of $\alpha \in S(d)$, and 0 otherwise.
- There is a nice class of functions of partioned permutations, called multiplicative. They are natural generating series of n-point functions.

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Theorem [Borot, Charbonnier, Garcia-Failde, Leid, Shadrin 21] – here only $g = 0$

Let $\phi, \phi^\vee$ be multiplicative functions related by $\phi = \zeta \star \phi^\vee$. Then their n-point functions $G_{0,n}$ and $G^\vee_{0,n}$ satisfy $G_{0,1}(X) = G^\vee_{0,1}(y)$, for $n = 2$ the [CMSS07]-relation (*) and for $n \geq 3$

$$G_{0,n}(X_1, \ldots, X_n) = \sum_{r_1,\ldots,r_n \geq 1} \prod_{i=1}^{n} \mathrm{O}^\vee_{r_i}(y_i) \sum_{T \in \mathcal{G}_{0,n}(\vec{r})} \tilde{\prod_{b \in \mathcal{B}(T)}} G^\vee_{0,|I(b)|}(y_{I(b)}) , \qquad X_i = \frac{y_i}{G^\vee_{0,1}(y_i)} ,$$

where $\mathcal{G}_{0,n}(\vec{r}) = \{$bipartite trees with n r_i-valent white vertices of operator weight $\mathrm{O}^\vee_{r_i}(y_i)\}$.

Notes

- The proof uses identities for weakly and strictly monotone Hurwitz numbers as well as Fock space techniques.
- The authors conjectured that the relation between G and $G^\vee$ also captures the x-y swap in topological recursion: Let $\omega^{(g)}_{,n}$ be meromorphic differentials computed from TR for spectral curve (x, y, B), and $\omega^{(g)\vee}_n$ be those for spectral curve (y, x, B). Setting
$$\omega^{(g)}_n(x_1, \dots, x_n) = G_{g,n}(x_1^{-1}, \dots, x_n^{-1}) \prod_{i=1}^n \frac{dx_i}{x_i}, \quad \omega^{(g)\vee}_n(y_1, \dots, y_n) = G^\vee_{g,n}(y_1, \dots, y_n) \prod_{i=1}^n \frac{dy_i}{y_i}$$
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then $G_{g,n}$, $G^\vee_{g,n}$ are related by previous formula.
- TR is built from $\omega_{0,1} = ydx$ from which symplectic form $dy \wedge dx$ is constructed. Several symplectic transformations were clarified by [Eynard, Orantin 07], but not the x-y swap.
- Conjecture proved by combined effort of [Hock 22] and [Alexandrov, Bychkov, Dunin-Barkowski, Kazarian, Shadrin 22].

Scalar quantum fields on noncommutative geometries

Euclidean QFT on NCG

- We adopt the point of view of Euclidean QFT which relates Schwinger functions to moments of a measure on a space of distributions.
- Linear theory governed by spectral dimension of Laplace-type operator.
- Non-linear QFT are challenging because multiplication of distributions needs work. Regularisation to finite-dimensional problem and careful limiting procedure are necessary.

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Aim is to learn how to build non-linear constructs of these distributions, defined by product in operator algebra. Works better than on manifolds!

Kontsevich matrix model with arbitrary potential

For $\lambda_1, \dots, \lambda_N > 0$ consider the Gaußian probability measure $d\mu_\Lambda(M)$ on $\mathcal{H}_N = \{\text{selfadjoint } N \times N\text{-matrices}\}$ with covariance $\langle M_{kl} M_{mn} \rangle = \frac{\delta_{kn}\delta_{lm}}{N(\lambda_k+\lambda_l)}$.

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Given a polynomial $V(x)$, we can deform the Gaußian measure to $d\mu_{\Lambda,V}(M) = \frac{1}{Z_{\Lambda,V}} e^{-N\,\mathrm{Tr}(V(M))} d\mu_\Lambda(M)$.

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- $\tau(t_1, t_3, t_5, ..) := \int_{\mathcal{H}_N} d\mu_{\Lambda,V}(M) \exp\Big(\sum_{k=0}^\infty t_{2k+1} \mathrm{Tr}(M^{2k+1})\Big)$ is a τ-function of the BKP integrable hierarchy [Borot, W 23]

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- $V(x) = \frac{\mathrm{i}}{3} x^3$ defines the original Kontsevich model. Then $Z_{\Lambda,\frac{\mathrm{i}}{3}x^3} =: Z(\vec{s})$ has asymptotic expansion in variables $s_{2k+1} = -(2k-1)!! \sum_{i=1}^N \lambda_i^{-(2k+1)}$
- $Z(\vec{s})$ is τ-function of KdV integrable hierarchy and generating series of intersection numbers of Chern classes on moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.

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- $Z(\vec{s})$ is τ-function of KdV integrable hierarchy and generating series of intersection numbers of Chern classes on moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.
- QFT motivates us to also study a quartic potential $V(x) = \frac{1}{4}x^4$.

Dyson-Schwinger equations and recursion

Dyson-Schwinger equations are identities between moments/cumulants obtained by integration by parts.

- Graphically, weight $\frac{1}{N}$ for edge, N for vertex and, by convention, N also for summation over face variables. Gives grading by Euler characteristic $\chi = \#_v - \#_e + \#_f$.
- DSE typically permit extension to complexified face labels $x \in \mathbb{C}$.

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Recursive structure of $1/N$-expansion

1. Non-linear equation for highest Euler characteristic. After change of variables $x = x(z)$ (sometimes) solvable by complex-analytic tools. Result defines $y(z)$.
2. Topological recursion [Eynard, Orantin 07] in decreasing Euler characteristic starting from x, y and Bergman kernel B which on $\mathbb{P}^1$ is $B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$.

The Kontsevich matrix model

Main definition

$$\Big(\int_{\mathcal{H}_N} d\mu_{\Lambda,\frac{\mathrm{i}}{3}x^3}(M)\; M_{a_1a_1}\cdots M_{a_na_n}\Big)_c - \delta_{n,1}\frac{N\lambda_{a_1}}{2} := \sum_{g=0}^{\infty} N^{2-n-2g}\, W^{(g)}_{a_1,\dots,a_n}$$

as formal power series, all a_i pairwise different, $(\ )_c$ stands for "cumulant".

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- $\big(W^{(0)}(x)\big)^2 = x^2 - \frac{2}{N}\sum_{k=1}^N \frac{W^{(0)}(\lambda_k)-W^{(0)}(x)}{\lambda_k - x}$ with solution [Makeenko, Semenoff 91] $W^{(0)}(x) = -\sqrt{x^2+c} + \frac{1}{N}\sum_{l=1}^N \frac{1}{\sqrt{\lambda_l^2+c}\,(\sqrt{x^2+c}+\sqrt{\lambda_l^2+c})}$ where $c = \frac{2}{N}\sum_{k=1}^N \frac{1}{\sqrt{\lambda_k^2+c}}$.

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- Suggests change of variables $z=\sqrt{x^2+c}$ that leads to $x(z)=z^2-c$ ramified at $z=0$ and $y(z) = z + \frac{1}{N}\sum_{k=1}^N \frac{1}{\varepsilon_k(\varepsilon_k - z)}$, $\quad \varepsilon_k = \sqrt{\lambda_k^2+c}$

Arrive at rational functions $\mathcal{W}^{(g)}(z_1,..,z_n) = W^{(g)}(x(z_1),..,x(z_n))$

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Theorem [Borot, Eynard, Orantin 15]

1. Given a spectral curve $(x, y : \Sigma \to \mathbb{P}^1, B)$, with technical assumptions.
2. Set $y =: \mathcal{W}_1^{(0)}$, $B(z,w) =: \mathcal{W}_2^{(0)}(z,w)dx(z)dx(w)$, $x^{-1}(x(z)) = \{\hat{z}^0 = z, \hat{z}^1, ..., \hat{z}^d\}$.
3. Assume that the following are holomorphic at any branch point of x:

$$L(x(z); z_2, ..., z_n) := \sum_{j=0}^{d} \mathcal{W}_n^{(g)}(\hat{z}^j, z_2, ..., z_n)$$

$$Q(x(z); z_2, ..., z_n) = \sum_{j=0}^{d} \Bigg(\sum_{\substack{I_1 \uplus I_2 = \{z_2,...,z_n\} \\ g_1+g_2=g}} \mathcal{W}_{|I_1|+1}^{(g_1)}(\hat{z}^j, I_1)\mathcal{W}_{|I_2|+1}^{(g_2)}(\hat{z}^j, I_2) + \mathcal{W}_{n+1,reg}^{(g-1)}(\hat{z}^j, \hat{z}^j, z_2, ..., z_n)\Bigg)$$

Linear and quadratic loop equations

Theorem [Borot, Eynard, Orantin 15]

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$$Q(x(z); z_2, \dots, z_n) = \sum_{j=0}^{d} \Bigg(\sum_{\substack{I_1 \uplus I_2 = \{z_2,\dots,z_n\} \\ g_1+g_2=g}} \mathcal{W}_{|I_1|+1}^{(g_1)}(\hat{z}^j, I_1)\mathcal{W}_{|I_2|+1}^{(g_2)}(\hat{z}^j, I_2) + \mathcal{W}_{n+1,reg}^{(g-1)}(\hat{z}^j, \hat{z}^j, z_2, \dots, z_n) \Bigg)$$

Then there is a formula which evaluates $\mathcal{W}_n^{(g)}$ in terms of $\mathcal{W}_{n'}^{(g')}$ with $2g'+n' < 2g+n$.

This formula is particularly simple (and universal!) under a projection property

$$\mathcal{W}_n^{(g)}(z, z_2, \dots, z_n) = \sum_{\beta=\text{ramif.pts}} \operatorname{Res}_{q=\beta} \Big(\int_\beta^q \mathcal{W}_2^{(0)}(z, .)dx(.) \Big) \mathcal{W}_n^{(g)}(q, z_2, \dots, z_n)dx(q)$$

(Otherwise 'blobbed topological recursion' [Borot, Shadrin 17])

Kontsevich model with quartic potential

- Take quartic potential $V(x) = \frac{1}{4}x^4$, consider 2-point function

$$\frac{1}{N}\int_{\mathcal{H}_N} d\mu_{\Lambda,\frac{1}{4}x^4}(M)\; M_{ab}M_{ba} =: \sum_{g=0}^{\infty} N^{-2g}\, G^{(g)}_{|ab|}$$

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Theorem [Grosse, W 09]

The planar two-point function satisfies the closed non-linear Dyson-Schwinger equation

$$\Big(x_1 + x_2 + \frac{1}{N}\sum_{k=1}^{N} G^{(0)}(x_1,\lambda_k)\Big)G^{(0)}(x_1,x_2) = 1 + \frac{1}{N}\sum_{k=1}^{N}\frac{G^{(0)}(\lambda_k,x_2) - G^{(0)}(x_1,x_2)}{\lambda_k - x_1}$$

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- Generalises to $N \to \infty$ and spectral dimension $\delta = \inf\{p : \sum_{i=1}^{\infty} \lambda_i^{-p/2} < \infty\}$. Renormalisation necessary and possible for $2[\frac{\delta}{2}] \in \{2, 4\}$
- 2D Moyal solved by [Panzer, W 18], general case of $2[\frac{\delta}{2}] \leq 4$ by [Grosse, Hock, W 19]

Since this talk is about TR, we focus on finite matrices.

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Theorem [Schürmann, W 19]

Ansatz $-x(-z) = \frac{1}{N}\sum_{k=1}^{N} G^{(0)}(x(z),\lambda_k) + x(z) + \frac{1}{N}\sum_{k=1}^{N} \frac{1}{\lambda_k - x(z)}$ and
$G^{(0)}(x(z_1),x(z_2)) =: \mathcal{G}^{(0)}(z_1,z_2)$

Solution for finite matrices

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$\mathcal{G}^{(0)}(z,w) = \dfrac{P_1^{(0)}(x(z),x(w))}{(x(z)+y(w))(x(w)+y(z))}$ where $\boxed{y(z) = -x(-z)}$ and

$$P_1^{(0)}(x(z),x(w)) = \frac{\prod_{u\in x^{-1}(\{x(w)\})}(x(z)+y(u))}{\prod_{k=1}^{d}(x(z)-x(\varepsilon_k))} \equiv P_1^{(0)}(x(w),x(z))$$

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Main definition [Branahl, Hock W 20]

For pairwise different $a_1,\dots,a_n$, set $W^{(g)}_{a_1,\dots,a_n} := [N^{2-2g-n}]\frac{(-1)^n\partial^n \log \mathcal{Z}_{\Lambda,\frac{1}{4}x^4}}{\partial\lambda_{a_1}\cdots\partial\lambda_{a_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(\lambda_{a_1}-\lambda_{a_2})^2}$ for $2g+n\geq 2$, and complexify to $\mathcal{W}^{(g)}_n(z_1,\dots,z_n)$. Moreover, $\mathcal{W}^{(0)}_1(z) = y(z)$.

Notes

- The linear and quadratic loop equations for $\mathcal{W}_n^{(g)}(z_1, \dots, z_n)$ hold locally [Hock, W 23]: $L(x(z); z_1, \dots, z_n)$ and $Q(x(z); z_1, \dots, z_n)$ are holomorphic at branch points of x.
  But the projection property does not hold: We have blobbed topological recursion.
- In [Hock, W 23] we established L, Q globally (i.e. exactly) for genus $g \leq 1$. Provides concrete recursion formulae for $\mathcal{W}_n^{(0)}(z_1, \dots, z_n)$ and $\mathcal{W}_n^{(1)}(z_1, \dots, z_n)$.

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- Found $x(z)$ for limit to 2D [Panzer, W 18] and 4D [Grosse, Hock, W 19] nc Moyal space:

$$D=2:\quad x(z)=z-\lambda\log(1+z)$$

$$D=4:\quad x(z)=z\cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda,1-\alpha_\lambda\\2\end{matrix}\middle|-z\right)\qquad \alpha_\lambda=\begin{cases}\dfrac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda|\leq\frac{1}{\pi}\\[1ex] \frac{1}{2}+\mathrm{i}\dfrac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda\geq\frac{1}{\pi}\end{cases}$$

- The latter encodes effective spectral dimension $\delta=4-\frac{2}{\pi}\arcsin(\lambda\pi)$. This dimension drop avoids the triviality problem of the usual $\lambda\phi^4_4$-model.

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Main message: expand about the solvable planar theory, not about the Gaußian!

Diversity

YAM — Young African mathematicians

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- five AIMS centers in Cameroon, Ghana, Rwanda, Senegal and South Africa

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Berlin Mathematics Research Center MATH+ — hcm HAUSDORFF CENTER FOR MATHEMATICS — STRUCTURES CLUSTER OF EXCELLENCE — MM Mathematics Münster Cluster of Excellence

- 10 fellows 2023/24 (5 female, 5 male)
- continues in October