Skew localizer for real index pairings

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Plan of the talk

- Skew Localizer
  - Review of complex index pairings and the spectral localizer
  - Periodic spectral localizer
  - Example of real index pairing (Kitaev chain)
  - Review of real index pairings
  - Skew localizer for real index pairings

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- Skew Localizer
  - Review of complex index pairings and the spectral localizer
  - Periodic spectral localizer
  - Example of real index pairing (Kitaev chain)
  - Review of real index pairings
  - Skew localizer for real index pairings
- Orientation flow
  - Review of real skew-adjoint Fredholm operators
  - Review of orientation flow in even component
  - Definition and properties of orientation flow in odd component
  - Example of orientation flow (half-sided Kitaev chain)

Complex index pairings

Fredholm operator T on separable Hilbert space $\mathcal{H}$ of type

$$T \;=\; PFP \;+\; \mathbf{1} - P\;,$$

where $P = P^2 = P^*$ orthogonal projection and F unitary such that $[F, P]$ compact

Theorem (Phillips 1997)

Let $t \in [0, 1] \mapsto H_t$ be a path of self-adjoint Fredholm operators, H_0 and H_1 invertible such that $H_t - H_0$ is compact for all t and $H_1 = F^* H_0 F$. If $P = \chi(H_0 \leq 0)$,

$$\mathrm{Ind}(PFP + \mathbf{1} - P) \;=\; \mathrm{Sf}(t \in [0, 1] \mapsto H_t)\;.$$

Spectral localizer for odd index pairings

A bounded invertible operator such that $U = A|A|^{-1}$

$D = D^*$ Dirac operator with D^{-1} compact such that
$[D, A]$ bounded and $E = \chi(D > 0)$ (unbounded Fredholm module)

Index pairing

$$T = EUE + \mathbf{1} - E$$

Spectral localizer (for $\kappa > 0$)

$$L_\kappa = \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix}$$

Restriction (Dirichlet) to finite-dimensional range of $\chi((|D| \oplus |D|) \leq \rho)$

$$L_{\kappa,\rho} = \begin{pmatrix} \kappa D_\rho & A_\rho \\ A_\rho^* & -\kappa D_\rho \end{pmatrix}$$

Clearly self-adjoint matrix:

$$(L_{\kappa,\rho})^* = L_{\kappa,\rho}$$

Spectral localizer for odd index pairings

Theorem (Loring, Schulz-Baldes 2017)

Given $D = D^*$ with compact resolvent and invertible A with invertibility gap $g = \|A^{-1}\|^{-1}$. Provided that

$$\|[D, A]\| \;\leq\; \frac{g^3}{12\,\|A\|\,\kappa} \tag{*}$$

and

$$\frac{2\,g}{\kappa} \;\leq\; \rho \tag{**}$$

the matrix $L_{\kappa,\rho}$ is invertible and

$$\tfrac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho}) \;=\; \mathrm{Ind}\big(EUE + \mathbf{1} - E\big)\;.$$

Spectral localizer for even index pairings

Index pairing

$$T = PFP + \mathbf{1} - P$$

Gapped Hamiltonian $H = H^*$ on $\mathcal{H}$ such that $P = \chi(H < 0)$

Dirac operator $D = D^*$ on $\mathcal{H} \oplus \mathcal{H}$ is odd w.r.t. grading $\Gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$

Thus $D = -\Gamma D \Gamma = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix}$ and Dirac phase $F = D_0 |D_0|^{-1}$

and $[H, D_0]$ bounded

Spectral localizer

$$L_\kappa \;=\; \begin{pmatrix} -H & \kappa\, D_0^* \\ \kappa\, D_0 & H \end{pmatrix} \;=\; -H \otimes \Gamma \;+\; \kappa\, D$$

Theorem (Loring, Schulz-Baldes 2020)

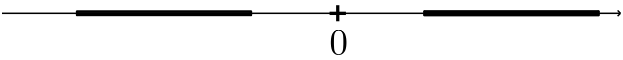
If $\|[H, D_0]\| < \infty$ and κ, ρ with (*) and (**), then

$$\mathrm{Ind}\big(PFP + \mathbf{1} - P\big) \;=\; \tfrac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho})\;.$$

Even periodic spectral localizer

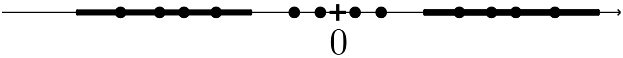
H finite range gapped periodic Hamiltonian on $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L$,

$\mathrm{spec}(H)$



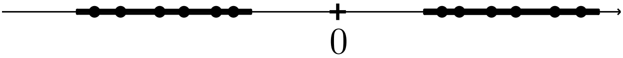
Dirichlet boundary condition:

$\mathrm{spec}(H_\rho)$



Periodic boundary condition:

$\mathrm{spec}(H_\rho^{\mathrm{per}})$



Even periodic spectral localizer

H finite range periodic Hamiltonian on $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L$,
$D = \sum_{j=1}^d X_j \gamma_j$ Dirac operator where $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\,\delta_{i,j}$

d even, thus

$$D = \begin{pmatrix} 0 & \sum_{j=1}^d X_j \widehat{\gamma}_j^* \\ \sum_{j=1}^d X_j \widehat{\gamma}_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix}$$

Index pairing

$$PFP + \mathbf{1} - P$$

for $P = \chi(H < 0)$ and $F = D_0 |D_0|^{-1}$

Theorem (Doll, Loring, Schulz-Baldes, submitted to Adv. Math.)

For ρ sufficiently large, κ sufficiently small

$$\mathrm{Ind}\big(PFP + \mathbf{1} - P\big) \;=\; \tfrac{1}{2}\,\mathrm{Sig}\left(\begin{pmatrix} -H_\rho^{\mathrm{per}} & \kappa\,(D_0^*)_\rho \\ \kappa\,(D_0)_\rho & H_\rho^{\mathrm{per}} \end{pmatrix}\right)$$

where ρ multiple of periods.

Even periodic spectral localizer

Periodic spectral localizer (for $\eta > 0$)

$$L^{\rm per}_{\eta,\rho} = \begin{pmatrix} \sum_{j=1}^d \left(\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)\right) & \sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j)\widehat{\gamma}_j^* \\ \sum_{j=1}^d \sin(\frac{\pi}{\rho} X_j)\widehat{\gamma}_j & -\sum_{j=1}^d \left(\mathbf{1} - \cos(\frac{\pi}{\rho} X_j)\right) \end{pmatrix} + \frac{1}{\eta} \begin{pmatrix} -H^{\rm per}_\rho & 0 \\ 0 & H^{\rm per}_\rho \end{pmatrix}$$

Theorem (Doll, Loring, Schulz-Baldes, submitted to Adv. Math.)

For

$$\rho \;\geq\; \frac{C\, d^4\, M\, \|H\|^3\, \eta^2}{g^6}\;, \tag{3}$$

where $g = \|H^{-1}\|^{-1}$, $M = \max_{j=1,\ldots,d} \|[X_j, H]\|$, *and* $C = 15 \cdot 10^6$ *and* $\eta \geq \frac{g}{4}$ *such that*

$$\left(1 - \frac{g}{\|H\|}\right)^2 + 4\left(1 - \frac{\eta}{\|H\|}\right) \;\leq\; \frac{g^2}{4\, d\, \eta\, \|H\|}\;. \tag{4}$$

the index of the index pairing is

$$\mathrm{Ind}(PFP + \mathbf{1} - P) \;=\; \tfrac{1}{2}\,\mathrm{Sig}(L^{\rm per}_{\eta,\rho}).$$

1d Bogoliubov de-Gennes Hamiltonian (class D)

Gapped Hamiltonian H acting on $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ such that $\overline{H} = -H$

Fermi projection $P = \chi(H \leq 0)$ such that $\overline{P} = \mathbf{1} - P$

Dirac operator $D = X \otimes \mathbf{1}$ with $X = \sum_{n \in \mathbb{Z}} n|n\rangle\langle n| - |0\rangle\langle 0|$ position op.

Hardy projection $E = \chi(D > 0)$ is real $E = \overline{E}$

Index pairing

$$T = E(\mathbf{1} - 2P)E + \mathbf{1} - E$$

$\mathrm{Ind}(T) = 0$ but well-defined is (Grossmann, Schulz-Baldes 2016)

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2$$

Alternative formula

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(EHE)) \bmod 2$$

Construction of skew localizer

Spectral localizer

$$L_\kappa = \begin{pmatrix} \kappa D & H \\ H & -\kappa D \end{pmatrix}$$

with $\overline{H} = -H$ and $\overline{D} = D$. Then $\mathrm{Sig}(L_{\kappa,\rho}) = 0$.

Use unitary basis change

$$R = \frac{1}{2}\begin{pmatrix} 1-\imath & 1+\imath \\ 1+\imath & 1-\imath \end{pmatrix}$$

then skew-localizer

$$\widehat{L}_\kappa = \imath R^* L_\kappa R = \begin{pmatrix} 0 & \imath H - \kappa D \\ \imath H + \kappa D & 0 \end{pmatrix}$$

and

$$\widehat{D} = \imath R^* \mathrm{diag}(D, -D) R = \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix}$$

Both real and skew-adjoint

Local formular for $\mathbb{Z}_2$-invariant

Restrictions to $\mathrm{Ran}(\chi(|\widehat{D}| \leq \rho))$

$$\widehat{L}_{\kappa,\rho} = \begin{pmatrix} 0 & \imath H_\rho - \kappa\, D_\rho \\ \imath H_\rho + \kappa\, D_\rho & 0 \end{pmatrix} \quad \text{and} \quad \widehat{D}_\rho = \begin{pmatrix} 0 & -D_\rho \\ D_\rho & 0 \end{pmatrix}$$

Real and skew-adjoint matrices

Theorem (Doll, Schulz-Baldes 2021)

For $\widehat{L}_{\kappa,\rho}$ and $\widehat{D}_\rho$ as above and κ, ρ with (*) and (**)

$$\begin{aligned} \mathrm{Ind}_2(E(\mathbf{1} - 2P)E + \mathbf{1} - E) &= \mathrm{sgn}\big(\mathrm{Pf}(\widehat{L}_{\kappa,\rho})\big)\mathrm{sgn}\big(\mathrm{Pf}(\widehat{D}_\rho)\big) \\ &= \mathrm{sgn}\big(\det(\imath H_\rho + \kappa\, D_\rho)\big)\mathrm{sgn}\big(\det(D_\rho)\big)\ . \end{aligned}$$

Example: Kitaev chain (in Majorana representation)

$$H = \imath \begin{pmatrix} 0 & V - \mu \\ -V^* + \mu & 0 \end{pmatrix}$$

acting on $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ with $V = \sum_{n \in \mathbb{Z}} |n+1\rangle\langle n|$ right shift, $\mu \in \mathbb{R}$

H gapped for $\mu \notin \{-1, 1\}$, $P = \chi(H \leq 0)$ such that $\overline{P} = \mathbf{1} - P$

Dirac operator $D = X \otimes \mathbf{1}$, Hardy projection $E = \chi(D > 0)$ real $E = \overline{E}$

$\mathrm{Ind}_2(T) = 1$ for $|\mu| < 1$ and $\mathrm{Ind}_2(T) = 0$ for $|\mu| > 1$

Theorem (Doll, Schulz-Baldes 2021)

For $\widehat{L}_{\kappa,\rho}$ and $\widehat{D}_\rho$ as above and κ, ρ with (*) and (**)

$$\mathrm{Ind}_2(E(\mathbf{1} - 2P)E + \mathbf{1} - E) \;=\; \mathrm{sgn}\left(\det\begin{pmatrix} \kappa X_\rho & -V_\rho + \mu \\ V_\rho^* - \mu & \kappa X_\rho \end{pmatrix}\right) .$$

Real index pairings

From now on $\mathcal{H}$ with complex conjugation $\mathcal{C}$. For operator A on $\mathcal{H}$

- complex conjugate: $\overline{A} = \mathcal{C}A\mathcal{C}$
- transpose: $A^T = (\overline{A})^*$

Implementation of symmetries:

Symmetry operator $\mathcal{S} = \overline{\mathcal{S}} = (\mathcal{S}^*)^{-1} \in \mathbb{B}(\mathcal{H})$ such that $\mathcal{S}^2 = \pm\mathbf{1}$

Symmetries of projection P:

- even/odd real: $\mathcal{S}^*\overline{P}\mathcal{S} = P$
- even/odd Lagrangian: $\mathcal{S}^*\overline{P}\mathcal{S} = \mathbf{1} - P$

Symmetries of unitary F in terms of a symmetry operator Σ:

- even/odd real: $\Sigma^*\overline{F}\Sigma = F$
- even/odd symmetric: $\Sigma^*F^T\Sigma = F$

Real index pairings

Index pairing

$$T \;=\; PFP + \mathbf{1} - P$$

For given combination of symmetries

- $\mathrm{Ind}(T) \in \mathbb{Z}$

or

- $\mathrm{Ind}(T) \in 2\mathbb{Z}$

or

- $\mathrm{Ind}(T) \;=\; 0$

In last case secondary invariant

$$\mathrm{Ind}_2(T) \;=\; \dim(\mathrm{Ker}(T)) \mod 2 \;\in\; \mathbb{Z}_2 \;,$$

may be well-defined.

Three types of index pairings

- projection P, unitary F: $\qquad T \;=\; PFP + \mathbf{1} - P$
- projections P, E:

$$T \;=\; P(\mathbf{1} - 2\,E)P + \mathbf{1} - P\,, \qquad T' \;=\; E(\mathbf{1} - 2\,P)E + \mathbf{1} - E \quad (1)$$

Proposition

For T and T' as in (1), $\mathrm{Ind}(T) = \mathrm{Ind}(T') = 0$ and $\mathrm{Ind}_2(T) = \mathrm{Ind}_2(T')$.

- unitaries U, F: set $P = \frac{1}{2}\begin{pmatrix} \mathbf{1} & -U \\ -U^* & \mathbf{1} \end{pmatrix}$ and $E = \frac{1}{2}\begin{pmatrix} \mathbf{1} & F \\ F^* & \mathbf{1} \end{pmatrix}$

$$T \;=\; P\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}P + \mathbf{1} - P\,, \qquad T' \;=\; E\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}E + \mathbf{1} - E \quad (2)$$

Proposition

For T and T' as in (2), $\mathrm{Ind}(T) = \mathrm{Ind}(T') = 0$ and $\mathrm{Ind}_2(T) = \mathrm{Ind}_2(T')$.

Index pairings with real symmetries (Kitaevs table)

$j \backslash d$		0	1	2	3	4	5	6	7
		$\Sigma^*\overline{F}\Sigma{=}F$ $\Sigma^2{=}\mathbf{1}$	$\Sigma^*\overline{E}\Sigma{=}E$ $\Sigma^2{=}\mathbf{1}$	$\Sigma^*F^T\Sigma{=}F$ $\Sigma^2{=}\mathbf{1}$	$\Sigma^*\overline{E}\Sigma{=}\mathbf{1}-E$ $\Sigma^2{=}-\mathbf{1}$	$\Sigma^*\overline{F}\Sigma{=}F$ $\Sigma^2{=}-\mathbf{1}$	$\Sigma^*\overline{E}\Sigma{=}E$ $\Sigma^2{=}-\mathbf{1}$	$\Sigma^*F^T\Sigma{=}F$ $\Sigma^2{=}-\mathbf{1}$	$\Sigma^*\overline{E}\Sigma{=}\mathbf{1}-E$ $\Sigma^2{=}\mathbf{1}$
0	$\mathcal{S}^*\overline{P}\mathcal{S}{=}P$ $\mathcal{S}^2{=}\mathbf{1}$	$\mathbb{Z}$	0	0	0	$2\,\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$\mathcal{S}^*\overline{U}\mathcal{S}{=}U$ $\mathcal{S}^2{=}\mathbf{1}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\,\mathbb{Z}$	0	$\mathbb{Z}_2$
2	$\mathcal{S}^*\overline{P}\mathcal{S}{=}\mathbf{1}-P$ $\mathcal{S}^2{=}\mathbf{1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\,\mathbb{Z}$	0
3	$\mathcal{S}^*U^T\mathcal{S}{=}U$ $\mathcal{S}^2{=}-\mathbf{1}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\,\mathbb{Z}$
4	$\mathcal{S}^*\overline{P}\mathcal{S}{=}P$ $\mathcal{S}^2{=}-\mathbf{1}$	$2\,\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
5	$\mathcal{S}^*\overline{U}\mathcal{S}{=}U$ $\mathcal{S}^2{=}-\mathbf{1}$	0	$2\,\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
6	$\mathcal{S}^*\overline{P}\mathcal{S}{=}\mathbf{1}-P$ $\mathcal{S}^2{=}-\mathbf{1}$	0	0	$2\,\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
7	$\mathcal{S}^*U^T\mathcal{S}{=}U$ $\mathcal{S}^2{=}\mathbf{1}$	0	0	0	$2\,\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Theorem (Doll, Schulz-Baldes 2021)

All $\mathbb{Z}_2$-indices can be calculated via skew localizer.

Skew localizer for real index pairings

Commuting symmetry operators $\mathcal{S}$ and Σ

Hamiltonian H such that $\mathcal{S}^*\overline{H}\mathcal{S} = \pm H$ and $[H, \Sigma] = 0$

Dirac operator $D = \binom{0\ D_0^*}{D_0\ 0}$ such that $\Sigma^*\overline{D_0}\Sigma = D_0^{(*)}$ and $[D_0, \mathcal{S}] = 0$

Spectral localizer

$$L_\kappa = \begin{pmatrix} -H & \kappa D_0^* \\ \kappa D_0 & H \end{pmatrix}$$

Even symmetry Q build from $\Sigma\mathcal{S}$ such that

$$Q^*\overline{L_\kappa}Q = -L_\kappa$$

Skew localizer for real index pairings

Unitary root R of Q such that

$$R^2 \;=\; Q\,, \qquad \overline{R} \;=\; R^*$$

Skew localizer

$$\widehat{L}_\kappa \;=\; \imath R^* L_\kappa R$$

real and skew-adjoint, and $\widehat{D} = \imath R^* D R$ real and skew-adjoint

Restriction to finite-dimensional range of $\chi(|\widehat{D}| \leq \rho)$: $\widehat{L}_{\kappa,\rho}$ and $\widehat{D}_\rho$

Theorem (Doll, Schulz-Baldes 2021)

If $\|[H, D_0]\| < \infty$ *and* κ, ρ *with* (*) *and* (**), *then*

$$\mathrm{Ind}_2(PFP + \mathbf{1} - P) \;=\; \mathrm{sgn}\big(\mathrm{Pf}(\widehat{L}_{\kappa,\rho})\big)\mathrm{sgn}\big(\mathrm{Pf}(\widehat{D}_\rho)\big).$$

Covers $T = P\begin{pmatrix} F & 0 \\ 0 & F\end{pmatrix}P + \mathbf{1} - P$ and $T = E(\mathbf{1} - 2P)E + \mathbf{1} - E$

Skew localizer for real index pairings

For pairing of two unitaries or two projections supplementary even chiral symmetry

$$I_\rho^* \widehat{L}_{\kappa,\rho} I_\rho \;=\; -\widehat{L}_{\kappa,\rho}\;, \qquad I_\rho^* \widehat{D}_\rho I_\rho \;=\; -\widehat{D}_\rho$$

In suitable basis

$$\widehat{L}_{\kappa,\rho} \;=\; \begin{pmatrix} 0 & -B_\rho^* \\ B_\rho & 0 \end{pmatrix}\;, \qquad \widehat{D}_\rho \;=\; \begin{pmatrix} 0 & -C_\rho^* \\ C_\rho & 0 \end{pmatrix}$$

Theorem (Doll, Schulz-Baldes 2021)

For κ, ρ with (*) and (**)

$$\mathrm{Ind}_2(PFP + \mathbf{1} - P) \;=\; \mathrm{sgn}\big(\det(B_\rho)\big)\mathrm{sgn}\big(\det(C_\rho)\big).$$

Main tool for proof: Orientation flow

Real skew-adjoint Fredholm operators

$\mathcal{H}$ separable Hilbert space with complex conjugation $\mathcal{C}$

Set of bounded real skew-adjoint Fredholm operators on $\mathcal{H}$

$$\mathbb{F}\mathbb{B}_{\text{skew}}(\mathcal{H}) \;=\; \{T \in \mathbb{B}(\mathcal{H}) : T = \overline{T} = -T^*, T \,\text{Fredholm}\}$$

where $\overline{T} = \mathcal{C}T\mathcal{C}$ complex conjugate of $T \in \mathbb{B}(\mathcal{H})$

Real skew-adjoint Fredholm operators

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$$\mathbb{F}\mathbb{B}_{\text{skew}}(\mathcal{H}) \;=\; \{T \in \mathbb{B}(\mathcal{H}) : T = \overline{T} = -T^*, T \,\text{Fredholm}\}$$

where $\overline{T} = \mathcal{C}T\mathcal{C}$ complex conjugate of $T \in \mathbb{B}(\mathcal{H})$

Connected components

- $\mathbb{F}\mathbb{B}_{\text{skew,ev}}(\mathcal{H}) \;=\; \{T \in \mathbb{F}\mathbb{B}_{\text{skew}}(\mathcal{H}) : \dim(\text{Ker}(T)) \in 2\mathbb{Z}\}$
- $\mathbb{F}\mathbb{B}_{\text{skew,od}}(\mathcal{H}) \;=\; \{T \in \mathbb{F}\mathbb{B}_{\text{skew}}(\mathcal{H}) : \dim(\text{Ker}(T)) + 1 \in 2\mathbb{Z}\}$

Fundamental group

$$\pi_1\big(\mathbb{F}\mathbb{B}_{\text{skew,ev}}(\mathcal{H})\big) \;=\; \pi_1\big(\mathbb{F}\mathbb{B}_{\text{skew,od}}(\mathcal{H})\big) \;=\; \mathbb{Z}_2$$

Real skew-adjoint Fredholm operators

$\mathcal{H}$ separable Hilbert space with complex conjugation $\mathcal{C}$

Set of bounded real skew-adjoint Fredholm operators on $\mathcal{H}$

$$\mathbb{FB}_{\rm skew}(\mathcal{H}) \;=\; \{T \in \mathbb{B}(\mathcal{H}) : T = \overline{T} = -T^*, T \,\text{Fredholm}\}$$

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Connected components

- $\mathbb{FB}_{\rm skew,ev}(\mathcal{H}) \;=\; \{T \in \mathbb{FB}_{\rm skew}(\mathcal{H}) : \dim(\mathrm{Ker}(T)) \in 2\mathbb{Z}\}$
- $\mathbb{FB}_{\rm skew,od}(\mathcal{H}) \;=\; \{T \in \mathbb{FB}_{\rm skew}(\mathcal{H}) : \dim(\mathrm{Ker}(T)) + 1 \in 2\mathbb{Z}\}$

Fundamental group

$$\pi_1\big(\mathbb{FB}_{\rm skew,ev}(\mathcal{H})\big) \;=\; \pi_1\big(\mathbb{FB}_{\rm skew,od}(\mathcal{H})\big) \;=\; \mathbb{Z}_2$$

Aim: Find explicit isomorphism $\pi_1\big(\mathbb{FB}_{\rm skew,od}(\mathcal{H})\big) \to \mathbb{Z}_2$

Review of orientation flow in even component

$t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\mathrm{skew,ev}}(\mathcal{H})$ with T_0 and T_1 invertible

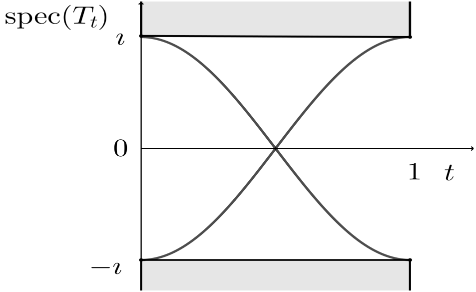
Orientation flow: $\quad \mathrm{Of}_2(t \in [0,1] \mapsto T_t) \ \in \mathbb{Z}_2$

Review of orientation flow in even component

$t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\text{skew,ev}}(\mathcal{H})$ with T_0 and T_1 invertible

Orientation flow: $\quad \mathrm{Of}_2(t \in [0,1] \mapsto T_t) \; \in \mathbb{Z}_2$

$T_t = -T_t^* \Rightarrow \mathrm{spec}(T_t) \in \imath\mathbb{R}, \qquad T_t = \overline{T_t} \Rightarrow \mathrm{spec}(T_t) = -\mathrm{spec}(T_t)$



Review of orientation flow in even component

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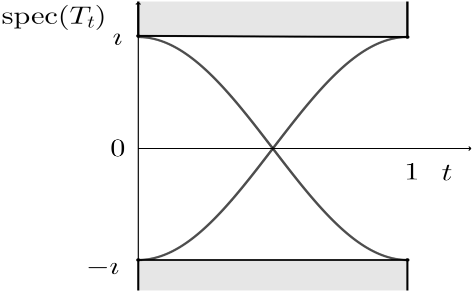

Theorem (Carey, Phillips, Schulz-Baldes 2018)

If $t \in [0,1] \mapsto T_t$ is analytic $\mathrm{Of}_2(t \in [0,1] \mapsto T_t)$ is modulo 2 equal to the sum of all eigenvalue crossings through 0 along the path, each one counted with its multiplicity.

Review of orientation flow in even component

Theorem (Carey, Phillips, Schulz-Baldes 2018)

Let $t \in [0,1] \mapsto T_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ be such that T_0 and T_1 are invertible

(i) *If T_t invertible for all $t \in [0,1]$ then* $\mathrm{Of}_2(t \in [0,1] \mapsto T_t) = 0$.

(ii) *For $t \in [1,2] \mapsto T_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ with T_2 invertible*

$$\mathrm{Of}_2(t \in [0,2] \mapsto T_t) \\ = \mathrm{Of}_2(t \in [0,1] \mapsto T_t) + \mathrm{Of}_2(t \in [1,2] \mapsto T_t) \bmod 2 .$$

(iii) *If $t \in [0,1] \mapsto T'_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ be such that $T_0 = T'_0$ and $T_1 = T'_1$ is connected to $t \in [0,1] \mapsto T_t$ via a norm-continuous homotopy in $\mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ leaving the endpoints fixed,*

$$\mathrm{Of}_2(t \in [0,1] \mapsto T_t) = \mathrm{Of}_2(t \in [0,1] \mapsto T'_t) .$$

Review of orientation flow in even component

Theorem (Carey, Phillips, Schulz-Baldes 2018)

Let $t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\text{skew}}(\mathcal{H})$ *be such that* T_0 *and* T_1 *are invertible*

(i) *If* T_t *invertible for all* $t \in [0,1]$ *then* $\mathrm{Of}_2(t \in [0,1] \mapsto T_t) = 0$.

(ii) *For* $t \in [1,2] \mapsto T_t \in \mathbb{FB}_{\text{skew}}(\mathcal{H})$ *with* T_2 *invertible*

$$\begin{aligned} &\mathrm{Of}_2(t \in [0,2] \mapsto T_t) \\ &\quad = \mathrm{Of}_2(t \in [0,1] \mapsto T_t) + \mathrm{Of}_2(t \in [1,2] \mapsto T_t) \bmod 2 \,. \end{aligned}$$

(iii) *If* $t \in [0,1] \mapsto T'_t \in \mathbb{FB}_{\text{skew}}(\mathcal{H})$ *be such that* $T_0 = T'_0$ *and* $T_1 = T'_1$ *is connected to* $t \in [0,1] \mapsto T_t$ *via a norm-continuous homotopy in* $\mathbb{FB}_{\text{skew}}(\mathcal{H})$ *leaving the endpoints fixed,*

$$\mathrm{Of}_2(t \in [0,1] \mapsto T_t) = \mathrm{Of}_2(t \in [0,1] \mapsto T'_t) \,.$$

Theorem (Carey, Phillips, Schulz-Baldes 2018)

Of_2 *induces isomorphism*

$$\mathrm{Of}_2 : \pi_1\left(\mathbb{FB}_{\text{skew,ev}}(\mathcal{H})\right) \to \mathbb{Z}_2$$

Index as Orientation flow

Theorem (Carey, Phillips, Schulz-Baldes 2018)

For Lagrangian P real F

$$\mathrm{Ind}_2(PFP + \mathbf{1} - P) = \mathrm{Of}_2(t \in [0,1] \mapsto (1-t)\imath(\mathbf{1} - 2P) + tF\imath(\mathbf{1} - 2P)F^*).$$

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Theorem (Carey, Phillips, Schulz-Baldes 2018)

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Proof of connection of $\mathbb{Z}_2$-index and Pfaffian of skew localizer:

Homotopy inveriance:

$$\mathrm{Ind}_2(PFP + \mathbf{1} - P) \;=\; \mathrm{Of}_2(t \in [0,1] \mapsto (1-t)\widehat{D} + t\widehat{L}_\kappa)$$

Restriction to finite volume:

$$\begin{aligned}\mathrm{Ind}_2(PFP + \mathbf{1} - P) \;&=\; \mathrm{Of}_2(t \in [0,1] \mapsto (1-t)\widehat{D}_\rho + t\widehat{L}_{\kappa,\rho}) \\ &=\; \mathrm{sgn}\big(\mathrm{Pf}(\widehat{L}_{\kappa,\rho})\big)\mathrm{sgn}\big(\mathrm{Pf}(\widehat{D}_\rho)\big)\end{aligned}$$

Motivation of orientation flow in odd component

For $t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\text{skew,od}}(\mathcal{H})$ with $\text{Ker}(T_0)$, $\text{Ker}(T_1)$ one-dimensional define orientation flow

$$\text{Of}'(t \in [0,1] \mapsto T_t) \ \in \mathbb{Z}_2$$

fulfilling

(i) concatenation (ii) homotopy invariance (iii) normalisation

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Concatenation: For $t \in [0,1] \mapsto T_t$ and $t \in [1,2] \mapsto T_t$ such that $\dim(\mathrm{Ker}(T_0)) = \dim(\mathrm{Ker}(T_1)) = \dim(\mathrm{Ker}(T_2)) = 1$

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Homotopy invariance: For $t \in [0,1] \mapsto T_t$ and $t \in [0,1] \mapsto T'_t$ with $T_0 = T'_0$ and $T_1 = T'_1$ having one-dimensional kernels connected via norm-continuous homotopy in $\mathbb{FB}_{\text{skew,od}}(\mathcal{H})$ leaving the endpoints fixed

$$\mathrm{Of}'(t \in [0,1] \mapsto T_t) \;=\; \mathrm{Of}'(t \in [0,1] \mapsto T'_t)$$

Motivation of orientation flow in odd component

Normalisation: For $t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\rm skew}(\mathcal{H})$ such that $t \in [0,1] \mapsto \mathrm{Ker}(T_t)$ is constant and one-dimensional

$$\mathrm{Of}'(t \in [0,1] \mapsto T_t) \;=\; 0$$

Motivation of orientation flow in odd component

Normalisation: For $t \in [0,1] \mapsto T_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ such that $t \in [0,1] \mapsto \mathrm{Ker}(T_t)$ is constant and one-dimensional

$$\mathrm{Of}'(t \in [0,1] \mapsto T_t) \;=\; 0$$

Orientation flow not fully determined

Proposition (Doll 2024)

Any path $t \in [0,1] \mapsto T_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ such that $\mathrm{Ker}(T_0) = \mathrm{Ker}(T_1)$ is one-dimensional is homotopic to a path $t \in [0,1] \mapsto T'_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ connecting $T_0 = T'_0$ to $T_1 = T'_1$ such that $\mathrm{Ker}(T_0) \subset \mathrm{Ker}(T'_t)$.

For $t \in [0,1] \mapsto T_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ such that $\mathrm{Ker}(T_0) = \mathrm{Ker}(T_1)$ is one-dimensional

$$\mathrm{Of}'(t \in [0,1] \mapsto T_t) \;=\; \mathrm{Of}(t \in [0,1] \mapsto P^{\perp}_{\mathrm{Ker}(T_0)} T_t P^{\perp}_{\mathrm{Ker}(T_0)})$$

Definition of orientation flow in odd component

Orientation flow w.r.t. real one-dimensional reference projection $P_{\text{ref}} \in \mathbb{B}(\mathcal{H})$ defined for paths $t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\text{skew}}(\mathcal{H})$ such that $\dim(\text{Ker}(T_0)) = \dim(\text{Ker}(T_1)) = 1$

$$\text{Of}_{P_{\text{ref}}}(t \in [0,1] \mapsto T_t) \;=\; \text{Of}(t \in [0,1] \mapsto P_{\text{ref}}^{\perp} T_t P_{\text{ref}}^{\perp})$$

if $P_{\text{Ker}(T_i)} \not\perp P_{\text{ref}}$ for $i = 0, 1$

Definition of orientation flow in odd component

Orientation flow w.r.t. real one-dimensional reference projection $P_{\mathrm{ref}} \in \mathbb{B}(\mathcal{H})$ defined for paths $t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\mathrm{skew}}(\mathcal{H})$ such that $\dim(\mathrm{Ker}(T_0)) = \dim(\mathrm{Ker}(T_1)) = 1$

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if $P_{\mathrm{Ker}(T_i)} \not\perp P_{\mathrm{ref}}$ for $i = 0, 1$

if $P_{\mathrm{Ker}(T_i)} \perp P_{\mathrm{ref}}$ set $O_i \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{O}\big(\mathrm{span}\big(\mathrm{Ran}(P_{\mathrm{ref}}), \mathrm{Ker}(T_i)\big)\big)$ and

$$t \in [0,1] \mapsto T_{0,t} \;=\; (1-t) O_0^* T_0 O_0 + t T_0 \qquad (i = 0)$$

$$t \in [0,1] \mapsto T_{1,t} \;=\; (1-t) T_1 + t O_1^* T_1 O_1 \qquad (i = 1)$$

Orientation flow w.r.t. P_{ref}:

$$\mathrm{Of}_{P_{\mathrm{ref}}}(t \in [0,1] \mapsto T_t) \;=\; \mathrm{Of}(t \in [0,1] \mapsto P_{\mathrm{ref}}^{\perp} (T_0 * T)_t P_{\mathrm{ref}}^{\perp}) \qquad (i = 0)$$

$$\mathrm{Of}_{P_{\mathrm{ref}}}(t \in [0,1] \mapsto T_t) \;=\; \mathrm{Of}(t \in [0,1] \mapsto P_{\mathrm{ref}}^{\perp} (T * T_1)_t P_{\mathrm{ref}}^{\perp}) \qquad (i = 1)$$

Definition of orientation flow in odd component

Definition

Let $t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\text{skew}}(\mathcal{H})$ be such that T_0 and T_1 have one-dimensional kernels and let $P_{\text{ref}} \in \mathbb{B}(\mathcal{H})$ be a real one-dimensional projection. Then the orientation flow of the path $t \in [0,1] \mapsto T_t$ w.r.t. P_{ref} is defined as

$$\mathrm{Of}_{P_{\text{ref}}}(t \in [0,1] \mapsto T_t) \;=\; \mathrm{Of}(t \in [0,1] \mapsto P_{\text{ref}}^{\perp} T'_t P_{\text{ref}}^{\perp})$$

where

$$T'_t \;=\; \begin{cases} T_t & \text{if } P_{\text{ref}} \not\perp P_{\mathrm{Ker}(T_0)}, P_{\text{ref}} \not\perp P_{\mathrm{Ker}(T_1)} \;, \\ (T_0 * T)_t & \text{if } P_{\text{ref}} \perp P_{\mathrm{Ker}(T_0)}, P_{\text{ref}} \not\perp P_{\mathrm{Ker}(T_1)} \;, \\ (T * T_1)_t & \text{if } P_{\text{ref}} \not\perp P_{\mathrm{Ker}(T_0)}, P_{\text{ref}} \perp P_{\mathrm{Ker}(T_1)} \;, \\ (T_0 * T * T_1)_t & \text{if } P_{\text{ref}} \perp P_{\mathrm{Ker}(T_0)}, P_{\text{ref}} \perp P_{\mathrm{Ker}(T_1)} \;. \end{cases}$$

Properties of orientation flow in odd component

Theorem (Doll 2024)

Let $t \in [0,1] \mapsto T_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ *fulfill* $\dim({\rm Ker}(T_0)) = \dim({\rm Ker}(T_1)) = 1$ *and let* $P_{\rm ref} \in \mathbb{B}(\mathcal{H})$ *be a real one-dimensional projection.*

(i) *If* $t \in [0,1] \mapsto {\rm Ker}(T_t)$ *is constant, then* ${\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_t) = 0$.

(ii) *For* $t \in [1,2] \mapsto T_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H})$ *with* $\dim({\rm Ker}(T_2)) = 1$

$$\begin{aligned} &{\rm Of}_{P_{\rm ref}}(t \in [0,2] \mapsto T_t) \\ &\qquad = {\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_t) + {\rm Of}_{P_{\rm ref}}(t \in [1,2] \mapsto T_t) \bmod 2 \,. \end{aligned}$$

(iii) ${\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_t) = {\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_{1-t})$.

(iv) ${\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_t) = {\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto -T_t)$.

(v) *If* $t \in [0,1] \mapsto T'_t \in \mathbb{F}\mathbb{B}_{\rm skew}(\mathcal{H}')$ *has invertible endpoints*

$$\begin{aligned} &{\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_t \oplus T'_t) \\ &\qquad = {\rm Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_t) + {\rm Of}(t \in [0,1] \mapsto T'_t) \bmod 2 \end{aligned}$$

Properties of orientation flow in odd component

Theorem (Doll 2024)

Let $t \in [0,1] \mapsto T_t$ and $t \in [0,1] \mapsto T'_t$ be such that $T_0 = T'_0$ and $T_1 = T'_1$ have one-dimensional kernels. If the paths are connected via a norm-continuous homotopy in $\mathbb{FB}_{\mathrm{skew}}(\mathcal{H})$ leaving the endpoints fixed,

$$\mathrm{Of}_{P_{\mathrm{ref}}}(t \in [0,1] \mapsto T_t) \;=\; \mathrm{Of}_{P_{\mathrm{ref}}}(t \in [0,1] \mapsto T'_t)\;.$$

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Theorem (Doll 2024)

For $t \in [0,1] \mapsto T_t \in \mathbb{FB}_{\rm skew}(\mathcal{H})$ such that $\mathrm{Ker}(T_0) = \mathrm{Ker}(T_1)$ is one-dimensional $\mathrm{Of}_{P_{\rm ref}}(t \in [0,1] \mapsto T_t)$ in independent of $P_{\rm ref} \in \mathbb{B}(\mathcal{H})$.

Properties of orientation flow in odd component

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Theorem (Doll 2024)

For any real one-dimensional projection $P_{\mathrm{ref}} \in \mathbb{B}(\mathcal{H})$

$$\mathrm{Of}_{P_{\mathrm{ref}}} : \pi_1\left(\mathbb{FB}_{\mathrm{skew,od}}(\mathcal{H})\right) \to \mathbb{Z}_2$$

is an isomorphism.

Orientation flow for half-sided Kitaev chain

Kitaev chain described by

$$H_0 \;=\; \frac{1}{2}\begin{pmatrix} V+V^* & \imath(V-V^*) \\ \imath(V-V^*) & -(V+V^*) \end{pmatrix} \in \mathbb{B}(\ell^2(\mathbb{Z})\otimes\mathbb{C}^2)$$

where $V=\sum_{m\in\mathbb{Z}}|m+1\rangle\langle m|$ right shift on $\ell^2(\mathbb{Z})$

Insertion of magnetic flux $2\pi\alpha$ in the cell between $n-1\in\mathbb{Z}$ and $n\in\mathbb{Z}$

$$H_{n,\alpha}=H_0+V_{n,\alpha}+V_{n,\alpha}^*, \qquad V_{n,\alpha}=|n\rangle\langle n-1|\otimes\frac{1}{2}\begin{pmatrix} e^{-\imath\pi\alpha}-1 & \imath(e^{\imath\pi\alpha}-1) \\ \imath(e^{-\imath\pi\alpha}-1) & -(e^{\imath\pi\alpha}-1) \end{pmatrix}$$

$$\text{PHS}: \qquad K^*\overline{H_{n,\alpha}}K \;=\; -H_{n,\alpha}, \quad K \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Cayley transformation $\kappa=\left(\begin{smallmatrix} 1 & -\imath \\ 1 & \imath \end{smallmatrix}\right)$

$$\alpha\in[0,1]\mapsto T_{n,\alpha} \;=\; \imath\kappa^* H_{n,\alpha}\kappa\in\mathbb{FB}_{\rm skew}(\ell^2(\mathbb{Z})\otimes\mathbb{C}^2)$$

Orientation flow for half-sided Kitaev chain

$$\alpha \in [0,1] \mapsto T_{n,\alpha} \;=\; \imath\kappa^* H_{n,\alpha}\kappa \in \mathbb{FB}_{\rm skew}(\ell^2(\mathbb{Z})\otimes\mathbb{C}^2)$$

Orientation flow

$$\mathrm{Of}(\alpha \in [0,1] \mapsto T_{n,\alpha}) \;=\; 1$$

Orientation flow for half-sided Kitaev chain

$$\alpha \in [0,1] \mapsto T_{n,\alpha} \;=\; \imath \kappa^* H_{n,\alpha} \kappa \in \mathbb{F}\mathbb{B}_{\mathrm{skew}}(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2)$$

Orientation flow

$$\mathrm{Of}(\alpha \in [0,1] \mapsto T_{n,\alpha}) \;=\; 1$$

Restriction $H_{r,n,\alpha}$ of $H_{n,\alpha}$ onto right half-line Hilbert space $\ell^2(\mathbb{N}) \otimes \mathbb{C}^2$

$$\alpha \in [0,1] \mapsto T_{r,n,\alpha} \;=\; \imath \kappa^* H_{r,n,\alpha} \kappa \in \mathbb{F}\mathbb{B}_{\mathrm{skew}}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^2)$$

$\mathrm{Ker}(T_{r,n,0}) = \mathrm{Ker}(T_{r,n,1})$ one-dimensional

Orientation flow for half-sided Kitaev chain

$$\alpha \in [0,1] \mapsto T_{r,n,\alpha} \;=\; \imath\kappa^* H_{r,n,\alpha}\kappa \in \mathbb{FB}_{\mathrm{skew}}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^2)$$

$\mathrm{Ker}(T_{r,n,0}) = \mathrm{Ker}(T_{r,n,1})$ one-dimensional

Proposition (Doll 2024)

Let $P_{\mathrm{ref}} \in \mathbb{B}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^2)$ be a real one-dimensional projection, then

$$\mathrm{Of}_{P_{\mathrm{ref}}}(\alpha \in [0,1] \mapsto T_{r,n,\alpha}) \;=\; \begin{cases} 0\,, & n \leq 1 \\ 1\,, & n \geq 2 \end{cases}.$$

Sketch of proof:

$\mathrm{Ker}(T_{r,n,0}) = \mathrm{Ker}(T_{r,n,1}) \;\Rightarrow\;$ Orientation flow independent of P_{ref}

$n \leq 1$: $\alpha \in [0,1] \mapsto T_{r,n,\alpha}$ constant $\;\Rightarrow\;$ Orientation flow vanishes

Proposition (Doll 2024)

Let $P_{\text{ref}} \in \mathbb{B}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^2)$ *be a real one-dimensional projection, then*

$$\mathrm{Of}_{P_{\text{ref}}}(\alpha \in [0,1] \mapsto T_{r,n,\alpha}) \;=\; \begin{cases} 0\,, & n \leq 1 \\ 1\,, & n \geq 2 \end{cases}\;.$$

Sketch of proof:

$n \geq 2$: $P_{\text{ref}} = P_{\text{Ker}(T_{r,n,0})}$

$$\widetilde{H}_{r,n,t} \;=\; \kappa((1-t)H_{r,n,0} + tH_{r,n,1})\kappa^*$$

$$= \begin{pmatrix} 0 & \sum_{m\geq 2} |m\rangle\langle m-1| \\ \sum_{m\geq 2} |m-1\rangle\langle m| & 0 \end{pmatrix} - 2t \begin{pmatrix} 0 & |n\rangle\langle n-1| \\ |n-1\rangle\langle n| & 0 \end{pmatrix}$$

One eigenvalue crossing

$$\widetilde{H}_{t,n,t} \begin{pmatrix} |n\rangle \\ \pm|n-1\rangle \end{pmatrix} \;=\; \pm(1-2t) \begin{pmatrix} |n\rangle \\ \pm|n-1\rangle \end{pmatrix}$$

$\Rightarrow \quad \mathrm{Of}_{P_{\text{ref}}}(\alpha \in [0,1] \mapsto T_{r,n,\alpha}) \;=\; 1$

Thank you for your attention!