Jean–Benoit Bost: "Foliations on algebraic varieties over number fields"

1. Foliations for Diophantine geometries
2. Algebraicity of leaves of foliations and Diophantine geometry
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1 Foliations

C^∞–foliations:

1. "foliated atlas": Let $d := \dim X$.

$$(\phi_i : U_i \to V_i)_{i \in I} \tag{1}$$

such that

$$\phi_j \circ \phi_i^{-1}(\Omega \times \{t_0\}) = \Omega' \times \{t'_0\}. \tag{2}$$

2. "integrable distributions": Let $F \hookrightarrow T_x$ subvector bundle, such that for all C^∞ local sections v_1, v_2 of F the commutator $[v_1, v_2]$ is also a section of F.

These two definitions are equivalent (sketch of proof):

$1 \Rightarrow 2$:

$$\begin{pmatrix} F_{U_i} & \hookrightarrow & T_{U_i} \\ \cong & (\text{via } D\phi_i) & \\ V_i \times (\mathbb{R}^f \oplus \{g\}) & \hookrightarrow & T_{V_i} = V_i \times \mathbb{R}^d \end{pmatrix} \tag{3}$$

$2 \Rightarrow 1$:

Choose C^∞–local coordinates $x_1, \ldots, x_d$ near P, a frame $v_1, \ldots, v_f$ of F near P , $v_i = \sum_{j=1}^{d} a_{ij} \frac{\partial}{\partial x_j}$, $1 \leq i \leq f$.

We may assume

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 & * & \dots & * \\ 0 & 1 & \dots & 0 & * & \dots & * \\ & & \dots & & \dots & & \\ 0 & 0 & \dots & 1 & * & \dots & * \end{pmatrix}, \tag{4}$$

so

$$v_1 = \frac{\partial}{\partial x_1} + * \frac{\partial}{\partial x_{f+1}} + \dots \tag{5}$$

$$\dots \tag{6}$$

$$v_f = \frac{\partial}{\partial x_f} + * \frac{\partial}{\partial x_{f+1}} + \dots . \tag{7}$$

Observations:

For all i, j, $[v_i, v_j]$ is a section of F, so $[v_i, v_j] = 0$. Hence:

$$\exp(tv_i) \circ \exp(tv_j) = \exp(tv_j) \circ \exp(tv_i). \tag{8}$$

Choose transversal $\phi_0 : T_0 \to V_0 \hookrightarrow \mathbb{R}^{d-f}$.

This gives

$$\phi : U \to] - \epsilon, \epsilon[^f \times V_0$$

via

$$\phi^{-1} : (t_1, \dots, t_f, u_1, \dots, u_{d-f}) \to exp(t_1 v_1) \circ \dots \circ exp(t_f v_f)(\phi_0(u_1, \dots, u_{d-f})).$$

2 Algebraic foliations

Let K be a field (often of characteristik 0), X a smooth algebraic variety over K, F a saturated coherent subsheaf of T_X closed under $[\ ,\ ]$. Let $\Delta \hookrightarrow X$ be closed of codimension ≥ 2 such that $F|_{X \setminus \Delta}$ is a subvectorbundle of $T_{X \setminus \Delta}$.

N.B.:

- If $\mathrm{rk}(F) = 1$ then $[\ ,\ ]$ is closed automatically.
- If $U \subset X$ is open dense an algebraic foliation on U extends uniquely to X.

Let now $\mathrm{char} K = 0$. Let $P \in (X \setminus \Delta)(K)$. Carrying over the proof of the first section we get the "formal leaf" of F through P:

$$\mathcal{F}_P \hookrightarrow \hat{X}_P,$$

it is a smooth formal subscheme of dimension f of the formal completion $\hat{X}_P$ of X at P. It can be thought of as the infinitesimal "integral" of the algebraic foliation.

Let $\sigma_0, \sigma_1 : K \hookrightarrow \mathbb{C}$ be two complex embeddings. Then we get two foliations $F_{\sigma_0}(\mathbb{C})$ and $F_{\sigma_1}(\mathbb{C})$ of $X_{\sigma_0}(\mathbb{C})$ respectively $X_{\sigma_1}(\mathbb{C})$.

Three "examples":

1) the "simplest" one: Let $K = \mathbb{C}$. Let $P, Q \in \mathbb{C}[X, Y]$ with no common factor, $X = \mathbb{A}^2$, $\Delta = (P = Q = 0)$. Let $F := \mathcal{O}_x(P\frac{\partial}{\partial X} + Q\frac{\partial}{\partial Y})$. This extends to $\mathbb{P}^2$.

2) Let G be a (smooth) algebraic group over K. Then $\mathrm{Lie}(G) = T_e G \cong \{\text{left } G\text{-invariant vector fields}\} = \Gamma(G, T_G)^G$.

Then a vector subspace $V \subset \mathrm{Lie}(G)$ corresponds to a G-invariant subvector bundle $F \hookrightarrow T_G$ and a Lie subalgebra $V \subset \mathrm{Lie}(G)$ to a G-invariant foliation.

When $K = \mathbb{C}$ there is the exponential

$$\exp_G : \mathrm{Lie}(G) \to G(\mathbb{C}),$$

the image of V under $\exp_G$ generates the leaf $\mathcal{F}$ of F through e.

If $G = \mathbb{G}_m$, then $\exp_G = \exp$. If G is an elliptic curve embedded in $\mathbb{P}^2$, then $\exp_G$ is given by $(\mathcal{P}, \mathcal{P}')$, where $\mathcal{P}$ is the Weierstra"s $\mathcal{P}$-function. If G is an abelian variety, then $\exp_G$ is given in terms of "abelian functions".

3) “linear foliations”:

Let S be a smooth algebraic variety over a field K, E a vector bundle over S, ∇ a connection on E. For a local section $s \in \Gamma(U, E)$ we have $\nabla s \in \Gamma(U, \Omega^1_X \otimes E)$ and the rule $\nabla(fs) = f\nabla s + df \otimes s$.

For $v \in \Gamma(U, T_X)$ we have the derivation of s in the direction of v: $\nabla_v s = \langle \nabla s, v\rangle \in \Gamma(U, E)$.

On a suitable open $U \subset X$ let's write $E \cong \mathcal{O}_U^{\oplus e}$. There

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_e \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_e \end{pmatrix} + A \begin{pmatrix} f_1 \\ \vdots \\ f_e \end{pmatrix},$$

where $A \in M_e(\Gamma(U, \Omega^1_X))$.

Let X be the total space of E and $p : X \to S$ the projection. Then the connection ∇ on E corresponds to an $F \hookrightarrow T_X$ such that $T_X = F \oplus T_p$.

Locally $X = S \times \mathbb{A}^e$, and $F_{(s,v)} \subset T_{(s,v)}(S \times \mathbb{A}^e) = T_s S \oplus K^e$ is given by

$$F_{(s,v)} := \{(\delta s, \delta v) \mid \delta s + (A(s) \cdot \delta s) \cdot \delta v = 0\}. \ ???$$

F is integrable (i.e. defines a foliation) if and only if ∇ is “integrable”, meaning that $[\nabla_{v_1}, \nabla_{v_2}] = \nabla_{[v_1,v_2]}$.

In this picture a formal section σ of E satisfying $\nabla\sigma = 0$ corresponds to a formal leaf of F.

3 Algebraicity of leaves of foliations

We use the following general setting:

- K a number field,
- X a smooth algebraic variety over K,
- $F \subset T_x$ a saturated coherent subsheaf of T_X, closed under $[\ ,\ ]$, $\Delta \subset X$ as in the previous section (i.e. F is an algebraic foliation),
- $P \in (X \backslash \Delta)(K)$ a rational point,
- $\mathfrak{F}_P \hookrightarrow \hat{X}_P$ the "formal leaf" at P.

We pose the question if there are "local"/arithmetic conditions on F which imply that $\mathcal{F}_P$ is algebraic.

We have the following results:

1.) "Schneider–Lang":

Suppose F is a rank 1 foliation, i. e. a foliation by curves, let

$X \hookrightarrow \mathbb{P}^N_K$ be an embedding and $\sigma_0 : K \hookrightarrow \mathbb{C}$ a complex embedding.

Theorem 3.1. *Let $f : \mathbb{C} \to X_{\sigma_0}(\mathbb{C})(\hookrightarrow \mathbb{P}^N(\mathbb{C}))$ be a nonconstant analytic mapping of finite order, s.t. there is a factorization $\hat{f}_0 : \hat{\mathbb{C}}_0 \to \mathcal{F}_P$.*

If $\mathcal{F}_P$ is not algebraic, then $f^{-1}((X\backslash\Delta)(K))$ is finite.

Comments:

1. "finite order ": Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be analytic. We say f is of finite order $:\Leftrightarrow \exists \vec{f} = (f_0, \ldots, f_N) : \mathbb{C} \to \mathbb{C}^{N+1}\backslash\{0\}$ analytic s.t. $f = (f_0, \ldots, f_N)$ and there exists $\rho > 0$ such that$||\vec{f}(\eta)|| = \mathcal{O}(e^{|\eta|^\rho})$. The order of f is defined to be the infimum of the possible ρ' s.

2. Nevanlinna theory:

   Let ω be the Fubini–Study 2–form on $\mathbb{P}^N(\mathbb{C})$. On $\mathbb{A}^N(\mathbb{C})$ we have

$$\begin{aligned}\omega = &\ \frac{1}{2\pi i}\partial\bar{\partial}\log(1+\sum_{i=1}^{N}|\eta_i|^2)^{-1} \\ \overset{\text{at } 0}{=} &\ \frac{1}{\pi}(dx_1\wedge dy_1+\cdots+dx_N\wedge dy_N).\end{aligned}$$

   Let $r>0$.

   Then the characteristic function (Nevanlinna, Ahlfors) is given by

$$T_f(r) := \int_{D(0,r)} \log^+ \frac{r}{|\eta|} f^*\omega.$$

   Fact: f is of finite order $\rho \Leftrightarrow \forall\epsilon>0\ T_f(r)=\mathcal{O}(r^{\rho+\epsilon})$.

   Proof of $\Rightarrow$:

$$f^*\omega = \frac{i}{2\pi}\partial\bar{\partial}\log||\vec{f}||^2$$

$$\begin{aligned}T_f(r) &= \int_{\mathbb{C}} \log^+ \frac{r}{|\eta|}(\frac{i}{2\pi}\partial\bar{\partial}\log||\vec{f}||^2) \\ &= \int_{\mathbb{C}}(\frac{i}{2\pi}\partial\bar{\partial}\log^+\frac{r^2}{|\eta|^2}\log||\vec{f}||) \quad \text{(Stokes, Green)} \\ &= \int_{\mu_r}\log||\vec{f}||-\log||\vec{f}||(0) \\ &= \mathcal{O}(r^\rho).\end{aligned}$$

   Note that $\frac{1}{2\pi i}\partial\bar{\partial}\log^+\frac{r^2}{|\eta|^2}=\delta_0-\mu_r$, where μ_r is defined by $\int_{\mathbb{C}}\phi\mu_r = \int_0^1\phi(re^{2\pi it})dt$.

Example: Assume $A \overset{i}{\hookrightarrow} \mathbb{P}^N(\mathbb{C})$ is an abelian variety.

Choose $v \in \mathrm{Lie}(A)$ and let

$$\begin{aligned} f : \mathbb{C} &\to A \hookrightarrow \mathbb{P}^N(\mathbb{C}) \\ t &\mapsto \exp_A(tv) = [tv]. \end{aligned}$$

Write $A \simeq \mathrm{Lie}(A)/\Gamma$.

Then

$$f^*\omega = ((t \mapsto tv)^* \circ \exp_A^* \circ i^*)\omega,$$

where $i^*\omega$ is a positive 2–form on A, $(\exp_A^* \circ i^*)\omega$ a translation invariant positive 2–form on $\mathrm{Lie}(A)$. Hence

$$T_f(r) \asymp \int_{\mathbb{C}} \log^+ \frac{r}{|\eta|} dx \wedge dy = r^2 \frac{\pi}{2}$$

has order 2.

3.1 Application to algebraic groups

Theorem 3.2. *(Lang): Assume given an algebraic group G over $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\mathrm{Lie}(G)$ the $\bar{\mathbb{Q}}$–Lie–algebra* $(\mathrm{Lie}(G))_{\mathbb{C}} \simeq \mathrm{Lie}(G)_{\mathbb{C}} \overset{\exp_{G_{\mathbb{C}}}}{\to} G(\mathbb{C})$.

1. Let $L \subset \mathrm{Lie}(G)$ line over $\bar{\mathbb{Q}}$.

   If $\exists v \in L_{\mathbb{C}} \backslash \{0\}$ such that $\exp_{G_{\mathbb{C}}}(v) \in G(\bar{\mathbb{Q}})$, then L is an algebraic Lie - subalgebra (i. e. $\exists H \hookrightarrow G$ algebraic subgroup over $\bar{\mathbb{Q}}$, $L = \mathrm{Lie}(H)$).

2. If for any $v \in \mathrm{Lie}(G) \backslash \{0\}$, $\exp_{G_{\mathbb{C}}}(v) \in G(\bar{\mathbb{Q}})$, then v is "unipotent", i.e. the map $t \mapsto \exp_{G(\mathbb{C})}(tv)$ is an algebraic map $\mathbb{G}_{a,\mathbb{C}} \to G_{\mathbb{C}}$.

   Proof:

   - $\mathbb{Q} \to K \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$,
     $\sigma_0 : K \to \mathbb{C}$,
     $X = G$, $F =$ transl. inv., $F_e = L$, $p = e$, $\mathfrak{F}_p = \exp L$,

     $$\begin{aligned} f : \mathbb{C} &\to G(\mathbb{C}) \\ t &\mapsto \exp_{G(\mathbb{C})}(tv) \end{aligned}$$

     of finite order, since $f^{-1}(G(K)) \supset \mathbb{Z}$ is infinite!

- apply 1. to $\tilde{G} = \mathbb{G}_a \times G$, $\tau = \frac{\partial}{\partial x} \oplus v$,
  $\Rightarrow$ graph $(t \to \exp(tv))$ is algebraic.

Exercises:

- $\mathbb{G}_a \times \mathbb{G}_a$ yields Hermite–Lindemann, i. e. $\beta \in \bar{\mathbb{Q}} \backslash \{0\} \Rightarrow e^\beta \notin \bar{\mathbb{Q}}$,
- $\mathbb{G}_m \times \mathbb{G}_m$ yields Gelfond–Schneider.

3.2 Conjecture of Grothendieck–Katz

Let X, $F \hookrightarrow T_X$ over K be as before. Choose models

$$\mathcal{X} \xrightarrow{\pi} \text{Spec } \mathcal{O}_K[\frac{1}{N}] \tag{9}$$

$$\mathcal{F} \hookrightarrow T_\pi \tag{10}$$

and a point $(0) \neq \mathfrak{p} \in \text{Spec } \mathcal{O}_K[\frac{1}{N}]$. We get the reductions $\mathcal{X}_{\mathbb{F}_\mathfrak{p}} \to \text{Spec } \mathbb{F}_\mathfrak{p}$ and

$\mathcal{F}_\mathfrak{p} = \mathcal{F}_{\mathbb{F}_\mathfrak{p}}$, an algebraic foliation over $\mathbb{F}_\mathfrak{p}$.

Let $p = \text{char } \mathbb{F}_\mathfrak{p}$. We have:

$$\mathcal{F}_\mathfrak{p} \text{ is } p\text{–closed } \Leftrightarrow \forall U \hookrightarrow \mathcal{X}_{\mathbb{F}_\mathfrak{p}}, \forall v \in \Gamma(U, \mathcal{F}_\mathfrak{p}), v^{[p]} := v \circ \cdots \circ v \in \Gamma(U, \mathcal{F}_\mathfrak{p}).$$

Fact: F is algebraically integrable $\Rightarrow$ for almost every p, $\mathcal{F}_\mathfrak{p}$ is p–closed (Grothendieck–Katz (GK), $\sim$ 1970).

$\Leftarrow$ was a conjecture, in general proven 1999 by Ekedahl, Shepherd–Barron, Taylor.

Partial results:

- Katz: linear systems, "Gau–Manin".
- diophantine approximation, André, J. B. B.

Theorem 3.3. *(J. B. B.) $P \in (X \backslash \Delta)(K)$, $\sigma_0 : K \hookrightarrow \mathbb{C}$. If F satisfies (GK), then the analytic leaf of F_{σ_0} through P is "Liouville" ($\hookrightarrow X_{\sigma_0}(\mathbb{C})$)(e. g. parametrized by analytic variety; $\Rightarrow$ $\mathcal{F}_\mathfrak{p}$ is algebraic).*

Application:

G algebraic group over K,

$$\mathcal{G} \to \mathrm{Spec}\mathcal{O}_K[\frac{1}{N}],$$

$(\mathrm{Lie}(\mathcal{G}))_K \cong \mathrm{Lie}(G)$; $(\mathrm{Lie}(\mathcal{G}))_{\mathbb{F}_\mathfrak{p}} \cong \mathrm{Lie}(\mathcal{G}_{\mathbb{F}_\mathfrak{p}})$.

Let $h \subset \mathrm{Lie}(G)$ be a Lie–subalgebra over K, then

h algebraic $\Leftrightarrow$ for almost all $\mathfrak{p}$, $h_{\mathbb{F}_\mathfrak{p}}$ is p–closed.

N.B.: $G = \mathbb{G}_m \times \mathbb{G}_m$, take for $h = K \cdot (x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y})$; is algebraic $\Leftrightarrow$ $\lambda \in \mathbb{Q}$, since $(\ )^{[p]} = x\frac{\partial}{\partial x} + \lambda^p y \frac{\partial}{\partial y}$ and

for almost all $\mathfrak{p}$, $h_\mathfrak{p}$ p–closed $\Leftrightarrow$ for almost all p, $[\lambda]^p = [\lambda]$ $\Leftrightarrow$ for almost all $p, [\lambda] \in \mathbb{F}_p$.

4 About the proofs:

Show formal leaf $\mathfrak{F}_P$ (of dimension 1 for simplicity) is algebraic:

Level 1: "Auxiliary polynomials":

$V \hookrightarrow \mathbb{C}^N$ any subset!

$$\begin{aligned} \eta_D : \mathbb{C}[X_1, \ldots, X_N]_{\leq D} &\to \mathbb{C}^V \\ p &\mapsto p|_V. \end{aligned}$$

Key observation: For $D \to \infty$, $\mathrm{rk}(\eta_D) \sim *D^{\dim \overline{V}^{\mathrm{Zar}}}$.

Proof:

$$\mathbb{C}[X_1, \ldots, X_N]_{\leq D} \cong \Gamma(\mathbb{P}^N, \mathcal{O}(D)) \stackrel{\eta_D}{\to} \mathbb{C}^V,$$
$$\mathbb{C}[X_1, \ldots, X_N]/I_+(\overline{V}) \underset{D \gg 0}{\simeq} \Gamma(\overline{V}^{\mathrm{Zar}}, \mathcal{O}(D)).$$

$\mathrm{rk}(\eta_D) = \dim \Gamma(\overline{V}^{\mathrm{Zar}}, \mathcal{O}(D))$ (Hilbert).

Extends to $\hat{V}$, the germ of the analytic or formal subvariety.

algebraicity $\Leftrightarrow$ $\mathrm{rk}(\eta_D) \sim *D^{\dim \hat{V}}$ $(\leq \dim \overline{\hat{V}})$.

Level 2: Algebraic varieties over function fields

Let C be a curve over k, $K = k(C)$, X projective variety over K, $P \in X(K)$ a rational point, $\hat{V} \subset \hat{X}_P$ smooth formal curve.

We choose again models $\mathfrak{X}$, $\hat{\mathcal{V}}$ and $\mathcal{P}$.

Proposition 4.1. *If $\hat{V}$ extends to a smooth formal subscheme $\hat{\mathcal{V}} \hookrightarrow \hat{\mathfrak{X}}_P$ and if $N := N_p\hat{\mathcal{V}}$ (line bundle) has positive degree, then $\hat{V}/\hat{\mathcal{V}}$ is algebraic.*

Proof:

Choose $\mathcal{O}(1)$ over $\mathfrak{X}$, consider

$$\begin{array}{cccc} \eta_D : & \Gamma(\mathfrak{X}, \mathcal{O}(D)) & \to & \Gamma(\hat{\mathcal{V}}, \mathcal{O}(D)) \\ & \| & & \downarrow \\ A^i D \subset & A_D & & \Gamma(\mathcal{V}_i, \mathcal{O}(D)) \end{array}$$

where $A^i_D := \{s \in A_D | \; s|_{\mathcal{V}_{i-1}} = 0\}$ and

$$\mathcal{V}_0 = \mathcal{P} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots .$$

Want $\mathrm{rk}(\eta_D) \sim *D^2$ for $D \to \infty$.

Observe

$$\begin{array}{ccccc} \eta_D(A^i_D)/\eta_D(A^{i+1}_D) & \hookrightarrow & \Gamma(\mathbb{P}, \check{N}^{\otimes i} \otimes \mathcal{O}(D)) & & \\ \dim_k \text{ — '' —} & \leq & \dim \text{ — '' —} & \leq & c(i + D) \\ & = & 0 \text{ if } \frac{c}{D} > \lambda & & . \end{array}$$

$\mathrm{rk}(\eta_D) \leq \sum_{i \geq 0} \dots$

Comments:

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$$\begin{array}{cccc} \eta_D : & \pi_*\mathcal{O}(D) & \to & \pi_*\mathcal{O}(D)|_{\hat{\mathcal{V}}} \\ & \| & & \\ & E_D & \text{v. b. over } C, & A_D = \Gamma(C, E_D) \\ & \cup & & \\ & E^i_D & \text{filtration} & \end{array}$$

- $E_D^i/E_D^{i+1} \overset{\eta_D^i}{\hookrightarrow} \mathcal{P}^*(\check{N}^{\otimes i} \otimes \mathcal{O}(D))$
- we assume wlog $\hat{V}$ Zariski–dense in X, want $d := \dim X = 1$.
  $\Rightarrow$ η_D injective.

Hence

$$\begin{aligned} 0 \underset{D \gg 0}{\leq} \deg E_D &= \sum_{i \geq 0} \deg E_D^i/E_D^{i+1} \\ &\leq \sum_{i \geq 0} \mathrm{rk}(E_D^i/E_D^{i+1})(-i \deg N + D \deg \mathcal{P}^*\mathcal{O}(1))^+ \\ &\sim *D^2. \end{aligned}$$

This implies $d = 1$.

Arakelov geometry:

$\mathfrak{X} \to \mathrm{Spec}\mathbb{Z}$ finite type, $\mathfrak{X}_{\mathbb{Q}}$ smooth.

Definition 4.2. *A hermitian vector bundle over $\mathfrak{X}$ is $\bar{E} = (E, ||.||)$ ($||.||$ C^∞–hermitian metric on $E_{\mathbb{C}}$, invariant under complex conjugation). Pullback tensor operation are defined as expected.*

For K a number field, $\mathfrak{X} = Spec\ \mathcal{O}_K$, $\mathfrak{X}(\mathbb{C}) = \{\sigma : K \hookrightarrow \mathbb{C}\}$, then a hermitian vector bundle is given by $\bar{E} = (E; ||.||_\sigma$ hermitian metric on E_σ for every $\sigma : K \hookrightarrow \mathbb{C})$.

Definition 4.3. *Arakelov degree: $\hat{\deg}\bar{E} := * \log \#(E/\mathcal{O}_K \cdot s) - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log ||s||_\sigma$ if $rk(E) = 1$, $s \in E \backslash \{0\}$, in general $\hat{\deg}\bar{E} = \hat{\deg} \wedge^{rk(E)} \bar{E}$.*

- $\bar{L_1}, \bar{L_2}$ of rk 1, $\hat{\deg}\bar{L_1} \otimes \bar{L_2} = \hat{\deg}\bar{L_1} + \hat{\deg}\bar{L_2}$,
- $0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$
  $\Rightarrow \hat{\deg}\bar{E} = \hat{\deg}\bar{S} + \hat{\deg}\bar{Q}$,
- $\bar{L_1}, \bar{L_2}$ rk 1, $\phi : L_{1,K} \to L_{2,K}$, $\phi \neq 0 \Rightarrow \hat{\deg}\bar{L_1} = \hat{\deg}\bar{L_2} + h(\phi)$,
  $h(\phi) := \sum_{\mathcal{P} \neq (0) \text{ in } \mathcal{O}_K} \log ||\phi||_{\mathcal{P}} + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log ||\phi||_\sigma$.

Level 3: Over a number field

K number field, X over K, $P \in X(K)$, $\hat{V} \subset \hat{X}_P$ formal curve.

Choose $\mathcal{O}(1)/X$.

Consider

$$\eta_{D,K} : \Gamma(X, \mathcal{O}(D)) \to \Gamma(\hat{V}, \mathcal{O}(D)),$$

$\hat{V}$ Zariski–dense in X, $\hat{V} = \underset{\to}{\lim} V_i$.

Let $N_K = T_P \hat{V}$.

Choose

$$\begin{array}{c} \mathfrak{X} \\ \pi \downarrow \uparrow \mathcal{P} \\ \text{Spec } \mathcal{O}_K \end{array} .$$

Let $\bar{E}_D = (\pi_* \mathcal{O}(D), ||.||_{L^2})$ and

$$\begin{array}{ccc} E_D^i & \subset & E_D \\ \ni & & \ni \\ s|_{V_{i-1}} = 0 & & s \end{array} ,$$

$$\eta^i_{D,K} : E^i_{D,K}/E^{i+1}_{D,K} \hookrightarrow \check{N}_K^{\otimes i} \otimes \mathcal{O}(D)|_{\mathcal{P}}.$$

As above

$$\begin{aligned} \hat{\deg} \bar{E}_D &= \sum_{i \geq 0} \hat{\deg}(\bar{E^i_D}/\bar{E^{i+1}_D}) \\ &= \sum_{i \geq 0} \mathrm{rk}(E^i_D/E^{i+1}_D)(-i \hat{\deg} \bar{N} + D \hat{\deg} \mathcal{P}^* \mathcal{O}(1) + h(\eta^i_D)). \end{aligned}$$

Basic principle: If $\frac{1}{D}(\dots)$ is $\ll 0$ when $\frac{i}{D} \gg 0$, this equality leads to a contradiction when $D \to \infty$ if $\dim X \geq 2$.

Observations:

$\hat{V}$ formal leaf $\Rightarrow$ $A^D i! \eta^i_D$ are defined over $\mathcal{O}_K$.

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\mathcal{P}}\log||\eta_D^i|| \leq i\log i + \gamma D,$$

$\hat{V}_\sigma$ analytic $\Rightarrow$ $||\eta_D^i||_\sigma \leq r^{-i}C^D$ "Cauchy",
$\log||\eta_D^i|| \leq \alpha D + \beta i.$

1. Schneider–Lang:

$$\mathbb{P}^N(\mathbb{C}) \hookleftarrow X,\ F,\ \sigma_0 : K \hookrightarrow \mathbb{C},\ f : \mathbb{C} \to X_{\sigma_0}(\mathbb{C}),$$

   $(\xi_\alpha)_{1\leq\alpha\leq k}$, $f(\xi_\alpha) = P_\alpha \in X(K)$, $\hat{V}_\alpha = \mathfrak{F}_{P_\alpha}$,

$$\eta_{D,K} : \Gamma(X, \mathcal{O}(D)) \to \oplus_{\alpha=1}^k \Gamma(\hat{V}_\alpha, \mathcal{O}(D)).$$

   Additional fact:

$$T_f(r) = \mathcal{O}(r^\rho) \Rightarrow \log||\eta_D^i|| \leq -\frac{k}{\rho} i \log\frac{i}{D} + c(i+D),$$
$$\frac{1}{D}h(\eta_D^i) \leq ([K:\mathbb{Q}] - \frac{k}{\rho})\frac{i}{D}\log\frac{i}{D} + c(\frac{i}{D}+1),$$

   gets very negative for $k \gg 0$.

2. Grothendieck–Katz:

   $X,\ F, \sigma_0$ as before, f (GK)

   Additional fact:

   (GK) formal leaf extends "almost smoothly" $\sum_{\mathcal{P}}\log||\eta_D^i|| \leq c(i+D)$ (weak smooth)

$$\forall\,\lambda > 0\,\exists c(\lambda) : \forall\,\lambda > 0\,\exists c(\lambda\log||\phi_D^i|| \leq -\lambda i + c(\lambda)D$$

$$\begin{aligned}-i\hat{\deg}\bar{N} + h(\eta_D^i) &\leq c'(i+D) + c(\lambda)D - \lambda i \\ &\leq -\epsilon i + c''D,\ \lambda > c'.\end{aligned}$$

Theorem 4.4. *(C. Gasbarri + J. B. B.):*

X, F rank 1, $P \in X(K)$, $\sigma_0 : K \hookrightarrow \mathbb{C}$, $\bar{\mathbb{Q}} \subset \mathbb{C}$,

$$f : C \to X_{\sigma_0}(\mathbb{C})$$

analytic along a leaf of F_{σ_0}, of finite order, C parabolic Riemann surface (affine algebraic curve).

$f(C)$ ist not algebraic

$$\Rightarrow \sum_{\xi \in C, f(\xi) \in (X \setminus \Delta)(\bar{\mathbb{Q}})} \frac{1}{[K(f(\xi)) : K]} < \infty.$$

Example: M affine curve over K, (E_1, ∇_1), (E_2, ∇_2).

Assume: $\exists \phi_{\mathbb{C}} : (E_{1,\mathbb{C}}, \nabla_{1,\mathbb{C}})^{\mathrm{an}} \cong (E_{2,\mathbb{C}}, \nabla_{2,\mathbb{C}})^{\mathrm{an}}$ (i.e. same monodromy).

If $\phi_{\mathbb{C}}$ is not algebraic

$$\Rightarrow \sum_{\substack{P \in C(\bar{\mathbb{Q}}), \\ \phi(P) \text{ defined over } \bar{\mathbb{Q}}}} \frac{1}{[K(P, \phi(P)) : K]} < \infty.$$