Higher symplectic stacks in diff. geometry

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Simplicial obj w/ Kan + Grothendieck pretop \rightarrow lie n-groupoids / Monta Equiv. \rightarrow lie n-algebroids

\text{group, 2-group}

differential n-stacks

tangent cx differentiation

integration
A simplicial manifold\(^1\) \(X_\bullet\) is a contravariant functor from \(\Delta\) the category of finite ordinals

\[
[0] = \{0\}, \quad [1] = \{0, 1\}, \quad \ldots, \quad [l] = \{0, 1, \ldots, l\}, \quad \ldots,
\]

with order-preserving maps to the category of manifolds.

\(^1\)Here, by manifolds we mean Banach manifolds as in [Lan95].
A simplicial manifold¹ $X_\bullet$ is a contravariant functor from $\Delta$ the category of finite ordinals

$$[0] = \{0\}, \quad [1] = \{0, 1\}, \quad \ldots, \quad [l] = \{0, 1, \ldots, l\}, \quad \ldots,$$

with order-preserving maps to the category of manifolds. More precisely, $X_\bullet$ consists of a tower of manifolds $X_l$, face maps $d_k^l : X_l \to X_{l-1}$ for $k = 0, \ldots, l$, and degeneracy maps $s_k^l : X_l \to X_{l+1}$. These maps satisfy the following simplicial identities

$$d_i^{l-1} d_j^l = d_{j-1}^{l-1} d_i^l \quad \text{if } i < j,$$

$$s_i^l s_j^{l-1} = s_{j+1}^l s_i^{l-1} \quad \text{if } i \leq j,$$

$$d_i^l s_j^{l-1} = \begin{cases} s_{j-1}^l d_i^{l-1} & \text{if } i < j, \\ \text{id} & \text{if } i = j, j + 1, \\ s_j^{l-2} d_{i-1}^{l-1} & \text{if } i > j + 1. \end{cases}$$

We drop the upper indices and just write $d_i$ and $s_i$ for simplicity when the context is clear.

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**Definition**

Let $\mathcal{C}$ be a category with coproducts and terminal object $\ast$. A **Grothendieck pretopology** on $\mathcal{C}$ is a collection $\mathcal{T}$ of arrows, called covers, with the following properties:

1. isomorphisms are covers;
2. the composite of two covers is a cover;
3. pull-backs of covers are covers; more precisely, for a cover $U \to X$ and an arrow $Y \to X$, the pull-back $Y \times_X U$ is representable and the canonical map $Y \times_X U \to Y$ is a cover;
4. for any object $X \in \mathcal{C}$, the map $X \to \ast$ to the terminal object is a cover;
Grothendieck pretopology

Similarly, we may have a simplicial object \(X\) in a general category \(C\).

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4. for any object \(X \in C\), the map \(X \to *\) to the terminal object is a cover;

A pretopology is called *subcanonical* if for any cover \(U \xrightarrow{\pi} X\) in \(\mathcal{T}\) and any morphism \(f \in C(U, M)\) such that \(pr_1^*f = pr_2^*f\), there is a unique \(\tilde{f} \in C(X, M)\) such that \(pr_1^*\tilde{f} = pr_2^*\tilde{f}\).
### Examples of Grothendieck pretopologies

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Exercise

Think of another example of a category with a Grothendieck pretoplogy.
simplicial simplexes and horns

Definition

Simplicial \(-\)-simplex \(\Delta[l]\) and the horn \(\Lambda[l, j]\):

\[
(\Delta[l])_k = \{ f : [k] \to [l] \mid f(i) \leq f(j), \forall i \leq j \},
\]
\[
(\Lambda[l, j])_k = \{ f \in (\Delta[l])_k \mid \{0, \ldots, j - 1, j + 1, \ldots, l\} \not\subseteq \{f(0), \ldots, f(k)\} \}.
\]

(2)
simplicial simplexes and horns

Definition

Simplicial $l$-simplex $\Delta[l]$ and the horn $\Lambda[l,j]$:

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$$(\Lambda[l,j])_k = \{ f \in (\Delta[l])_k \mid \{0,\ldots,j - 1,j + 1,\ldots,l\} \notin \{f(0),\ldots,f(k)\}\}.$$ (2)

$\Lambda[l,j]$ is a simplicial set obtained from the simplicial $l$-simplex $\Delta[l]$ by taking away its unique non-degenerate $l$-simplex as well as the $j$-th of its $l + 1$ non-degenerate $(l - 1)$-simplices:

\[
\begin{align*}
\Lambda[1,1] &\quad \Lambda[1,0] &\quad \Lambda[2,2] &\quad \Lambda[2,1] &\quad \Lambda[2,0] &\quad \Lambda[3,3] &\quad \Lambda[3,2] \\
. &\quad . &\quad . &\quad . &\quad . &\quad . &\quad .
\end{align*}
\]
Lie $n$-groupoids

**Definition**

A *$n$-groupoid object* [Get09, Hen08, Zhu09] in a category with a Grothendieck pretopology $(\mathcal{C}, \mathcal{T})$ is a simplicial object $X_\bullet$ in $\mathcal{C}$ where the natural projections

$$p_j^l : X_l = \text{hom}(\Delta[l], X_\bullet) \to \text{hom}(\Lambda[l,j], X_\bullet) =: X_{l,j}$$

are covers for all $1 \leq l \geq j \geq 0$ and isomorphisms for all $0 \leq j \leq l > n$. It is further a *$n$-group* object if $X_0 = \ast$—the terminal object.
A \textit{n-groupoid object} [Get09, Hen08, Zhu09] in a category with a Grothendieck pretopology \((C, T)\) is a simplicial object \(X_\bullet\) in \(C\) where the natural projections

\[ p^l_j : X_l = \text{hom}(\Delta[l], X_\bullet) \to \text{hom}(\Lambda[l,j], X_\bullet) =: X_{l,j} \quad (3) \]

are covers for all \(1 \leq l \geq j \geq 0\) and isomorphisms for all \(0 \leq j \leq l > n\). It is further a \textit{n-group} object if \(X_0 = \ast\)–the terminal object.

Take \((C, T)\) to be \((\text{Mfd}, T_{\text{subm}})\), then we have \textbf{Lie n-groupoids}.
Remark

*The notation* \( \text{hom}(S, X) \), *when* \( S \) *is a simplicial set and* \( X \) *is a simplicial object in a category* \( C \), *is a certain limit* [Hen08, Section 2].  *E.g.*

\[
\text{hom}(\Delta[m], X) = X_m, \quad \text{hom}(\Lambda[2, 1], X_\bullet) = X_1 \times_{d_1, X_0, d_0} X_1.
\]

*However it is not obvious that the limit* \( \text{hom}(\Lambda[m, j], X) \) *as a presheaf is representible*, *and it is a result of* [Hen08, Corollary 2.5].
Example: group

When \( n = 1 \), \( X_0 = * \), a Lie \( n \)-groupoid \( X_\bullet = NG \) for a Lie group \( G \).

\[ X_0 = *, \ X_i = G \times^i, \ d_0(g) = d_1(g) = * \] and

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d_0(g_1, \ldots, g_i) = (g_2, \ldots, g_i), \quad d_j(g_1, \ldots, g_i) = (g_1, \ldots, g_jg_{j+1}, \ldots, g_i),
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d_i(g_1, \ldots, g_i) = (g_1, \ldots, g_{i-1}), \quad s_j(g_1, \ldots, g_i) = (g_1, \ldots, g_j, 1_{s(g_j)}, g_{j+1}, \ldots, g_i).
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Example: group

When $n = 1$, $X_0 = \ast$, a Lie $n$-groupoid $X_\bullet = NG$ for a Lie group $G$. $X_0 = \ast$, $X_i = G^{\times i}$, $d_0(g) = d_1(g) = \ast$ and

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The existence of a composition for arrows is given by the condition Kan$(2, 1)$, whereas the composition of an arrow with the inverse of another is given by Kan$(2, 0)$ and Kan$(2, 2)$.

\[ \text{Kan}(2,2) \quad \text{Kan}(2,1) \quad \text{Kan}(2,0) \]

(4) Chenchang Zhu (Mathematisches Institut, Göttingen) Higher symplectic stacks in diff. geometry
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When $n = 1$, composition and inverse are unique since we have $\text{Kan}(2, j)$.\(\text{(4)}\)
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![Diagram](image-url)
Exercise

*Associativity is given by Kan(2, 1)! and Kan(3, 2).*
A very nice conceptual way: 2-group $G_\bullet = G_1 \Rightarrow G_0$ is a categorification of the concept of group [BL04]: $G_1 \Rightarrow G_0$ is a category equipped with a multiplication functor $m: G_\bullet \times G_\bullet \to G_\bullet$, an identity functor $e : \ast \to G_\bullet$, an inverse functor $i : G_\bullet \to G_\bullet$. 

Exercise

_Associativity is given by Kan(2, 1)! and Kan(3, 2)._
exercise and intro of 2-group

Exercise

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**Associativity is given by Kan(2, 1)! and Kan(3, 2).**

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**Trivial example:** A group $G$ is a 2-group with $G_0 = G$, $G_1 = G$. 

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Assumptions are given by Kan(2, 1)! and Kan(3, 2).

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Trivial example: A group $G$ is a 2-group with $G_0 = G$, $G_1 = G$.

Example

$BS^1$ with $G_0 = \ast$, $G_1 = S^1$ is a 2-group, equipped with $m_1 = m_{S^1}: S^1 \times S^1 \rightarrow S^1$. It’s a groupoid functor because $S^1$ is abelian. This corresponds to the group structure on $PU(H)$. 

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$O(n) \xrightarrow{\sim} So(n) \xrightarrow{\sim} Spin(n) \xrightarrow{\sim} String(n)$
String 2-group

\[ O(n) \rightarrow So(n) \rightarrow Spin(n) \rightarrow String(n) \]

*String*(\(n\)) may be viewed as a central extension of 2-groups

\[ 1 \rightarrow BS^1 \rightarrow String(n) \rightarrow Spin(n) \rightarrow 1, \]

given by the generator of

\[ H^2(BSpin(n), BS^1) = H^3(BSpin(n), S^1) = H^4(BSpin(n), \mathbb{Z}) = \mathbb{Z}. \]
When $n > 1$, the composition of two arrows is in general not unique, but any two of them can be joined by a 2-morphism $h$ given by Kan(3, 2).
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Thus the Kan condition corresponds to composition and inverse, up to homotopy.
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The $n$-th homotopy is unique. Thus when $n = 2$, and $X_0 = \ast$, it gives [Zhu09] us a (weak) Lie 2-group in the sense of categorification by Baez-Lauda [BL04].
The associativity is given by $\text{Kan}(3, 2)$ and $\text{Kan}!(2, 1)$.

$$\text{Kan}(3, 2) \quad \text{Kan}!(2, 1)$$

Proof of associativity.

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Updated Route map

Differentiable n-stacks

Lie n-groupoids/Monta Equiv

Group, 2-group

Simplicial obj w/ Kan + Grothendieck pretop

Symp + Differentiable n-stacks

m-shifted symp Lie n-groupoids/Symp

(BG, ω) Group, 2-group,
several interesting models

m-shifted symp Lie n-algebroids

Tangent ex differentiation
Integration

Chenchang Zhu (Mathematisches Institut) Higher symplectic stacks in diff. geom.
Morita Equivalence

Lie $n$-groupoids/Morita equivalences $=$ differentiable $n$-stacks (geometric stack in diff. geom.)
Lie $n$-groupoids/Morita equivalences $=$ differentiable $n$-stacks (geometric stack in diff. geom.) Morita equivalence may be realised as

- higher bibundles (just as Morita equivalence between $C^*$-algebras may be realised as Hilbert bimodules);
- zig-zag of hypercovers;
- zig-zag of weak equivalences.
Let us first recall a Morita bibundle for Lie groupoids. $E$ is a Morita bibundle iff $E \to X_0$ is a right $Y_\bullet$ principal bundle, $E \to Y_0$ is a left $X_\bullet$ principal bundle, and $X_\bullet Y_\bullet$ actions commute:

\[\begin{array}{ccc}
X_1 & \to & E \\
\downarrow & & \downarrow \\
X_0 & \to & Y_0 \\
\end{array}\]
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X_1 & \rightarrow & E & \rightarrow & Y_1 \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
X_0 & \rightarrow & E & \rightarrow & Y_0
\end{array}
$$

A usual Lie groupoid morphism $X_\bullet \xrightarrow{f_\bullet} Y_\bullet$ is represented by the bibundle $Bd(f) = X_0 \times_{f_0,Y_0,t} Y_1$. 
This leads to a bicolored Kan fibration $M_{1,-1} = X_1$, $M_{0,0} = E$, $M_{-1,1} = Y_1$, $M_{2,-1} = X_2$, $M_{1,0} = X_1 \times X_0 E$, $M_{0,1} = E \times \times Y_0 Y_1$, $M_{-1,2} = Y_2$.

Exercise

What are higher $M_{\bullet,\bullet}$’s in this case?
For an bicolored simplicial object $M_{\bullet,\bullet}$ in $(C, T)$, i.e. a simplicial object $M$ together with a map $M \to \Delta[1]$, bicolored (strict) Kan conditions are defined as following:

\[
\text{Kan}(p, q; k) : \text{Hom}(\Delta^{p,q}, M) \to \text{Hom}(\wedge_{k}^{p,q}, M) \text{ is a cover,}
\]

\[
\text{Kan}(p, q; k)! : \text{Hom}(\Delta^{p,q}, M) \to \text{Hom}(\wedge_{k}^{p,q}, M) \text{ is an isomorphism,}
\]

for $p, q \in \{-1, 0, 1, \ldots, \}$ and $k \in \{0, 1, \ldots, p + q + 1\}$. 

(7)
Bibundles

For an bicolored simplicial object $M_{\bullet,\bullet}$ in $(C, T)$, i.e. a simplicial object $M$ together with a map $M \to \Delta[1]$, bicolored (strict) Kan conditions are defined as following:

$$\text{Kan}(p, q; k) : \text{Hom}(\Delta^{p,q}, M) \to \text{Hom}(\wedge^p_k M)$$ is a cover,

$$\text{Kan}(p, q; k)! : \text{Hom}(\Delta^{p,q}, M) \to \text{Hom}(\wedge^p_k M)$$ is an isomorphism, \hspace{1cm} (7)

for $p, q \in \{-1, 0, 1, \ldots, \}$ and $k \in \{0, 1, \ldots, p + q + 1\}$.

$\Delta^{p,q}$ is $\Delta[p + q + 1]$ equipped with the following map to $\Delta[1]$:

$$0, \ldots, p \mapsto 0, \hspace{0.5cm} p + 1, \ldots p + q + 1 \mapsto 1.$$

Similarly for $\wedge^p_k M$. 

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Higher symplectic stacks in diff. geom.
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for $p, q \in \{-1, 0, 1, \ldots, \}$ and $k \in \{0, 1, \ldots, p + q + 1\}$.

If $M_{\bullet,\bullet}$ satisfies all Kan($p, q; k$) and Kan($p, q; k)!$ when $p + q \geq n + 1$, then $X_\bullet := M_{\bullet,-1}$ and $Y_\bullet := M_{-1,\bullet}$ are $n$-groupoid objects, and $M_{\geq 0, \geq 0}$ is a Morita bibundle between them [BZ] inspired by [Lur09].
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\text{Kan}(p, q; k) : \text{Hom}(\Delta^{p,q}, M) \rightarrow \text{Hom}(\wedge_k^{p,q}, M) \text{ is a cover}, \\
\text{Kan}(p, q; k)! : \text{Hom}(\Delta^{p,q}, M) \rightarrow \text{Hom}(\wedge_k^{p,q}, M) \text{ is an isomorphism},
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for $p, q \in \{-1, 0, 1, \ldots,\}$ and $k \in \{0, 1, \ldots, p + q + 1\}$.

If $M \bullet \bullet$ satisfies all $\text{Kan}(p, q; k)$ and $\text{Kan}(p, q; k)!$ when $p + q \geq n + 1$, then $X \bullet := M \bullet _{-1}$ and $Y \bullet := M_{-1} \bullet$ are $n$-groupoid objects, and $M_{\geq 0, \geq 0}$ is a Morita bibundle between them [BZ] inspired by [Lur09].

Composition of bibundles is provided by the right adjoint of subdivision.
Bibundles

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\text{Kan}(p, q; k)! : \text{Hom}(\Delta^{p,q}, M) \to \text{Hom}(\wedge_k^{p,q}, M) \text{ is an isomorphism,}
\]
for $p, q \in \{-1, 0, 1, \ldots, \}$ and $k \in \{0, 1, \ldots, p + q + 1\}$.  \(\text{(7)}\)

Exercise

What is $Bd(f)$ in this case for a morphism $f : X_\bullet \to Y_\bullet$? Will it satisfies all Kan conditions?
Simplicial boundary of an \( i \)-simplex is a simplicial set
\[
(\partial \Delta[i])_k = \{ f \in (\Delta[i])_k \mid \{0, \ldots, i\} \not\subseteq \{f(0), \ldots, f(k)\} \},
\]
The boundary space of a simplicial object \( X_\bullet \) is
\[
\partial_i(X_\bullet) = \text{Hom}(\partial \Delta[i], X_\bullet) = \{(x_0, \cdots, x_i) \in X_{i-1} \times_i^{i+1} \mid d_a(x_b) = d_{b-1}(x_a) \forall a < b\}
\]
(8)
Hypercovers

Simplicial boundary of an $i$-simplex is a simplicial set

$$(\partial \Delta[i])_k = \{ f \in (\Delta[i])_k \mid \{0, \ldots, i\} \not\subset \{ f(0), \ldots, f(k) \} \},$$

The boundary space of a simplicial object $X_\bullet$ is

$$\partial_i(X_\bullet) = \text{Hom}(\partial \Delta[i], X_\bullet) = \{(x_0, \cdots, x_i) \in X_{i-1}^{i+1} \mid d_a(x_b) = d_{b-1}(x_a) \forall a < b \}.$$ (8)

If $X_\bullet$, $Y_\bullet$ are two Lie $n$-groupoids, then a simplicial morphism $f_\bullet : X_\bullet \to Y_\bullet$ is called a hypercover if the maps

$$q_i := (\{d_0, \cdots, d_i\}, f_i) : X_i \to \partial_i(X_\bullet) \times \partial_i(Y_\bullet) Y_i = \text{Hom}(\partial \Delta[i] \to \Delta[i], X_\bullet \to Y_\bullet).$$ (9)

are surjective submersions for $0 \leq i < n$ and an isomorphism for $i = n$. Then $q_i$ is automatically an isomorphism for $i \geq n + 1$ [Zhu09, Lemma 2.5]. Two Lie $n$-groupoids $X_\bullet$ and $Y_\bullet$ are called Morita equivalent if and only if there is a third Lie $n$-groupoid $Z_\bullet$ with hypercovers $f_\bullet : Z_\bullet \to X_\bullet$ and $g_\bullet : Z_\bullet \to Y_\bullet$. 
Hypercovers

Simplicial boundary of an \(i\)-simplex is a simplicial set

\[
(\partial \Delta[i])_k = \{ f \in (\Delta[i])_k \mid \{0, \ldots, i\} \not\subset \{f(0), \ldots, f(k)\}\},
\]

The boundary space of a simplicial object \(X\) is

\[
\partial_i(X) = \text{Hom}(\partial \Delta[i], X) = \{(x_0, \cdots, x_i) \in X_{i-1} \times |d_a(x_b) = d_{b-1}(x_a) \forall a < b\}
\]

If \(X, Y\) are two Lie \(n\)-groupoids, then a simplicial morphism \(f : X \to Y\) is called a hypercover if the maps

\[
q_i := ((d_0, \cdots, d_i), f_i) : X_i \to \partial_i(X) \times \partial_i(Y) Y_i = \text{Hom}(\partial \Delta[i] \to \Delta[i], X \to Y)
\]

are surjective submersions for \(0 \leq i < n\) and an isomorphism for \(i = n\). Then \(q_i\) is automatically an isomorphism for \(i \geq n + 1\) [Zhu09, Lemma 2.5]. Two Lie \(n\)-groupoids \(X\) and \(Y\) are called Morita equivalent if and only if there is a third Lie \(n\)-groupoid \(Z\) with hypercovers \(f : Z \to X\) and \(g : Z \to Y\). This works for \(n\)-groupoid objects in \((\mathcal{C}, \mathcal{T})\) replacing surjective submersions by covers.
In general, $\partial_i(X_\bullet)$ might not be a manifold any more, even if $X_\bullet$ is a Lie $n$-groupoid, and the most-right-hand-side of (8) is a set-theoretical description for it. Nevertheless, the right-hand-side of (??) is always a manifold if $f_\bullet$ is a hypercover—even though it seems to be logically dependent, this is something one can prove level-wise inductively [Zhu09, Sect.2.1] just like the case of horn spaces mentioned in Remark 5. Therefore hypercover is a well-defined concept for Lie $n$-groupoids.

Remark

Hypercovers for Lie 1-groupoid is called (strong) Morita morphism or strong equivalence some times.
Weak equivalences

Similar to a hypercover, a simplicial morphism $f_\bullet : X_\bullet \rightarrow Y_\bullet$ is called a weak equivalence [BG15, RZ20] if

$$r_i : \text{Hom}(\Delta[i] \rightarrow \Delta[i+1], X_\bullet \rightarrow Y_\bullet) \rightarrow \text{Hom}(\partial\Delta[i] \rightarrow \wedge[i+1, i+1], X_\bullet \rightarrow Y_\bullet)$$

is a surjective submersion for $i < n$ and isomorphism for $i \geq n$. This explicit formula comes from the fact that

$$X_\bullet \xrightarrow{\text{w.eq.}} Y_\bullet \quad \text{iff} \quad X_\bullet \xleftarrow{\text{h.c.}} X_\bullet \times Y_\bullet Y'_\bullet \xrightarrow{\text{h.c.}} Y_\bullet$$
Weak equivalences

Similar to a hypercover, a simplicial morphism \( f_\bullet : X_\bullet \to Y_\bullet \) is called a weak equivalence [BG15, RZ20] if

\[
r_i : \text{Hom}(\Delta[i] \to \Delta[i+1], X_\bullet \to Y_\bullet) \to \text{Hom}(\partial \Delta[i] \to \Lambda[i+1, i+1], X_\bullet \to Y_\bullet)
\]
is a surjective submersion for \( i < n \) and isomorphism for \( i \geq n \). This explicit formula comes from the fact that

\[
X_\bullet \xrightarrow{\text{w.eq.}} Y_\bullet \quad \text{iff} \quad X_\bullet \xleftarrow{\text{h.c.}} X_\bullet \times Y_\bullet \quad Y'_\bullet \xrightarrow{\text{h.c.}} Y_\bullet
\]

Remark

Weak equivalence for Lie groupoids is called weak equivalence too.
1. higher bibundles (just as Morita equivalence between $C^*$-algebras may be realised as Hilbert bimodules);
2. zig-zag of hypercovers;
3. zig-zag of weak equivalences.
1. **higher bibundles** (just as Morita equivalence between $C^*$-algebras may be realised as Hilbert bimodules);

2. **zig-zag of hypercovers**;

3. **zig-zag of weak equivalences**.

$2 \iff 3$ is proven in [RZ20]

$1 \iff 2$ will be proven in [BZ]
1 higher bibundles (just as Morita equivalence between $C^*$-algebras may be realised as Hilbert bimodules);
2 zig-zag of hypercovers;
3 zig-zag of weak equivalences.

$2 \iff 3$ is proven in [RZ20]
$1 \iff 2$ will be proven in [BZ]

In fact, $X_\bullet \xrightarrow{f} Y_\bullet$ is a weak equivalence iff the bundlisation $Bd(f)$ is a Morita bibundle. The condition required in weak equivalence is exactly the last “missing” Kan condition $\text{Kan}(n, 0; n + 1)$ for $M \to \Delta[1]$. 
A simplicial manifold $X_\bullet$ has a natural tangent simplicial vector bundle

$$\ldots T X_2|_{X_0} \cong T X_1|_{X_0} \Rightarrow T X_0,$$

where $X_0$ is a submanifold in $X_l$ via iterated degeneracy $X_0 \xrightarrow{s_0} X_1 \xrightarrow{s_0} X_2 \ldots \xrightarrow{s_0} X_l$. The face and degeneracy maps are simply $Td_i$'s and $Ts_i$'s.
A simplicial manifold $X_\bullet$ has a natural tangent simplicial vector bundle

\[ \cdots TX_2|_{X_0} \to TX_1|_{X_0} \to TX_0, \quad (10) \]

where $X_0$ is a submanifold in $X_l$ via iterated degeneracy $X_0 \overset{s_0}{\to} X_1 \overset{s_0}{\to} X_2 \cdots \overset{s_0}{\to} X_l$. The face and degeneracy maps are simply $Td_i$'s and $Ts_i$'s.

Applying Dold-Kan to (10), we obtain the following,

\[
\mathcal{T}_l X = \begin{cases} 
\ker TP_i|_{X_0} = \cap_{i=0}^{l-1} \ker d_i & \text{for } l > 0, \\
TX_0 & \text{for } l = 0, \\
0 & \text{for } l < 0,
\end{cases}
\]

and

\[ \partial := (-1)^l Td_i. \]
A simplicial manifold $X_\bullet$ has a natural tangent simplicial vector bundle

$$\ldots TX_2|_{X_0} \to TX_1|_{X_0} \to TX_0,$$

where $X_0$ is a submanifold in $X_l$ via iterated degeneracy $X_0 \xrightarrow{s_0} X_1 \xrightarrow{s_0} X_2 \ldots \xrightarrow{s_0} X_l$. The face and degeneracy maps are simply $Td_i$'s and $Ts_i$'s.

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and $\partial := (-1)^l Td_i$.

If $X_\bullet$ is further a Lie $n$-groupoid, since the horn projection (3) is a surjective submersion, $\ker TP_i$ is a vector bundle. The complex of vector bundles $(\mathcal{T}X, \partial)$ is defined to be the tangent complex of $X_\bullet$.  

Chenchang Zhu (Mathematisches Institut, Göttingen) 
Higher symplectic stacks in diff. geom.
A simplicial manifold $X_\bullet$ has a natural tangent simplicial vector bundle

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(10)

where $X_0$ is a submanifold in $X_l$ via iterated degeneracy

$$X_0 \xrightarrow{s_0} X_1 \xrightarrow{s_0} X_2 \ldots \xrightarrow{s_0} X_l.$$

The face and degeneracy maps are simply $Td_i$’s and $Ts_i$’s.

Applying Dold-Kan to (10), we obtain the following,

$$T_l X = \begin{cases} 
\ker TP_l|_{X_0} = \cap_{i=0}^{l-1} \ker d_i & \text{for } l > 0, \\
TX_0 & \text{for } l = 0, \\
0 & \text{for } l < 0,
\end{cases}
$$

and $\partial := (-1)^l Td_i$.

If $X_\bullet$ is further a Lie $n$-groupoid, since the horn projection (3) is a surjective submersion, $\ker TP_l$ is a vector bundle. The complex of vector bundles $(T_\bullet X, \partial)$ is defined to be the tangent complex of $X_\bullet$. Clearly, $T_\bullet$ concentrated on degree $0, \ldots, n$. 

Chenchang Zhu (Mathematisches Institut, Göttingen)
Higher symplectic stacks in diff. geom
\[ X_\bullet \xrightarrow{\sim} Y_\bullet \iff T_\bullet X \xrightarrow{\sim} T_\bullet Y \]
Tangent complex, Morita Equivalence, $L_\infty$-algebroids

\[ X_\bullet \xrightarrow{\sim} Y_\bullet \quad \Rightarrow \quad \mathcal{T}_\bullet X \xrightarrow{\sim} \mathcal{T}_\bullet Y \]

As sketched in [Šev] and further being verified in [?], $\mathcal{T}_\bullet X$ further carries a split Lie $n$-algebroid structure.
Tangent complex, Morita Equivalence, $L_\infty$-algebroids

\[ X_\bullet \xrightarrow{\sim, M.E.} Y_\bullet \quad \implies \quad T_\bullet X \xrightarrow{\sim, W.Eq.} T_\bullet Y \]

As sketched in [Šev] and further being verified in [?], $T_\bullet X$ further carries a split Lie $n$-algebroid structure.

**Definition**

An *N-manifold* $\mathcal{M}$ consists of a pair as follows:

- a topological space $M$ (the “body”)
- a sheaf $\mathcal{O}_M$ over $M$ of graded commutative algebras, locally for an open set $U \subset M$, $\mathcal{O}(U) = C^\infty(U) \otimes S^\bullet(V_\bullet^*)$ (the sheaf of “functions”), where $V_\bullet = \bigoplus_{i < 0} V_i$ a finite dimensional $\mathbb{Z}_{<0}$-graded vector space.

*Vector fields* on $\mathcal{M}$ are derivations of $\mathcal{O}(M)$. An **NQ-manifold** is an N-manifold equipped with a *homological vector field*, i.e. a degree 1 vector field $Q$ such that $[Q, Q] = 0$. 
A graded vector bundle $\bigoplus E_{i<0}$ over $M \leadsto$ an N-manifold $\mathcal{M}$ with

$$\mathcal{O}(M) = C^\infty(M) \oplus S^\bullet(\bigoplus_{i<0}\Gamma(E_i^*))$$.
A graded vector bundle $\bigoplus E_{i<0}$ over $M \xrightarrow{\sim} \text{an N-manifold } \mathcal{M}$ with

$$\mathcal{O}(\mathcal{M}) = C^\infty(M) \oplus S^\bullet(\bigoplus_{i<0} \Gamma(E_i^*))$$.

It turns out that all N-manifolds are of such a type. But one needs to fix some connections to arrive at an associated graded vector bundle [BP13].
$L_{\infty}$ algebroids

A graded vector bundle $\bigoplus E_{i<0}$ over $M \hookrightarrow$ an $N$-manifold $\mathcal{M}$ with

$$\mathcal{O}(\mathcal{M}) = C^\infty(M) \oplus S^\bullet(\bigoplus_{i<0} \Gamma(E_i^*))$$

It turns out that all $N$-manifolds are of such a type. But one needs to fix some connections to arrive at an associated graded vector bundle [BP13].

If the local model $V_\bullet = \bigoplus_{-n \leq i < 0} V_i$, the $NQ$ manifold $\mathcal{M}$ is called an $NQ$-$n$-manifold, or a Lie $n$-algebroid.
$L_\infty$ algebroids

A graded vector bundle $\bigoplus E_i < 0$ over $M \rightsquigarrow$ an N-manifold $\mathcal{M}$ with

$$\mathcal{O}(\mathcal{M}) = C^\infty(M) \bigoplus S^\bullet(\bigoplus_{i < 0} \Gamma(E^*_i)).$$

It turns out that all N-manifolds are of such a type. But one needs to fix some connections to arrive at an associated graded vector bundle [BP13].

If the local model $V_\bullet = \bigoplus_{-n \leq i < 0} V_i$, the NQ manifold $\mathcal{M}$ is called an NQ-$n$-manifold, or a Lie $n$-algebroid.

E.g., when $n = 1$, $\mathcal{M} = A[1]^2$, $A \to M$ is a vector bundle, equipped with vector bundle morphism $\rho : A \to TM$ called anchor, and Lie bracket $l_2 : \Gamma(\bigwedge^2 A) \to \Gamma(A)$ satisfying:

$$l_2(X, fY) = fl_2(X, Y) + \rho(X)(f)Y.$$

This is exactly what a Lie algebroid is.

$^2A[1]$ concentrates on degree -1, since $V[k]_l = V_{k+l}$.
In general, for an NQ-$n$-manifold $\mathcal{M}$, we pick an associated graded vector bundle $\bigoplus_{n \leq i < 0} A^i \to M$, then $Q$ gives rise to an anchor $\rho : A_{-1} \to TM$, and brackets $l_k : \Gamma(\wedge^k A_\bullet) \to A_\bullet$ of degree 1, via

\[
\begin{align*}
Q(f) &= \rho^*(df), \\
Q(\xi^k)(X^{i_1}_1, \ldots, X^{i_j}_j) &= -\langle \xi^k, l_j(X^{i_1}_1, \ldots, X^{i_j}_j) \rangle, \quad j \neq 2, \\
Q(\xi^k)(X^{-1}_1, X^{-k}_2) &= \rho(X^{-1}_1)\langle \xi^k, X^{-k}_2 \rangle - \langle \xi^k, l_2(X^{-1}_1, X^{-k}_2) \rangle, \\
Q(\xi^1)(X^{-1}_1, X^{-1}_2) &= \rho(X^{-1}_1)\langle \xi^1, X^{-1}_2 \rangle - \rho(X^{-1}_2)\langle \xi^1, X^{-1}_1 \rangle \\
&\quad - \langle \xi^1, l_2(X^{-1}_1, X^{-1}_2) \rangle,
\end{align*}
\]

for all $f \in C^\infty(M), \xi^k \in \Gamma(A^*_{-k}), X^{i_p}_p \in \Gamma(A_{i_p}), \quad -i_1 - \cdots - i_j = -k + 1 - j$. 

(11)
In general, for an NQ-\(n\)-manifold \(M\), we pick an associated graded vector bundle \(\bigoplus_{n \leq i < 0} A_i \rightarrow M\), then \(Q\) gives rise to an anchor \(\rho : A_{-1} \rightarrow TM\), and brackets \(l_k : \Gamma(\wedge^k A_\bullet) \rightarrow A_\bullet\) of degree 1, via

\[
\begin{align*}
Q(f) &= \rho^*(df), \\
Q(\xi^k)(X_1^{i_1}, \ldots, X_j^{i_j}) &= -\langle \xi^k, l_j(X_1^{i_1}, \ldots, X_j^{i_j}) \rangle, \quad j \neq 2, \\
Q(\xi^k)(X_1^{-1}, X_2^{-k}) &= \rho(X_1^{-1})\langle \xi^k, X_2^{-k} \rangle - \langle \xi^k, l_2(X_1^{-1}, X_2^{-k}) \rangle, \\
Q(\xi^1)(X_1^{-1}, X_2^{-1}) &= \rho(X_1^{-1})\langle \xi^1, X_2^{-1} \rangle - \rho(X_2^{-1})\langle \xi^1, X_1^{-1} \rangle - \langle \xi^1, l_2(X_1^{-1}, X_2^{-1}) \rangle,
\end{align*}
\]

for all \(f \in C^\infty(M), \xi^k \in \Gamma(A^*_{-k}), X_p^{i_p} \in \Gamma(A_{i_p}), -i_1 - \cdots - i_j = -k + 1 - j\).

Such an \(n\)-term graded vector bundle \(A_\bullet\) is a split Lie \(n\)-algebroid.
**Definition (split Lie $n$-algebroid [SZ])**

A split Lie $n$-algebroid (There is a [1] shift here comparing to the original definition.) is a non-positively graded vector bundle $\mathcal{A} = A_{-1} \oplus \cdots \oplus A_{-n}$ over a manifold $M$ equipped with a bundle map $\rho : A_{-1} \to TM$ (called the anchor), and $n + 1$ many brackets $l_i : \Gamma(\wedge^i \mathcal{A}) \to \Gamma(\mathcal{A})$ with degree 1 for $1 \leq i \leq n + 1$, such that

1. 
   \[ \sum_{i+j=k+1} \sum_{\sigma \in Sh_{i,k-i}} \pm l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), X_{\sigma(i+1)}, \cdots, X_{\sigma(k)}) = 0, \]  
   \[ (12) \]

2. $l_2$ satisfies the Leibniz rule with respect to $\rho$: 
   \[ l_2(X^0, fX) = fl_2(X^0, X) + \rho(X^0)(f)X, \quad \forall X^0 \in \Gamma(A_0), f \in C^\infty(M), X \in \Gamma(A) \]

3. For $i \neq 2$, $l_i$'s are $C^\infty(M)$-linear.
Example (Poisson manifolds $\subset$ Lie 1-algebroids)

A Poisson manifold $(P, \pi)$ equips with a bivector field $\pi \in \Gamma(\wedge^2 TP)$, with $[\pi, \pi] = 0$, or equivalently a Poisson bracket $\{,\}$ on $C^\infty(P)$ which is Lie and Leibniz. We then have a Lie 1-algebroid $T^*P$ with anchor $\pi : T^*P \to TP$, and Lie bracket $l_2(df, dg) = d\{f, g\}$. 
Example (Courant algebroids)

An exact Courant algebroid \((\mathcal{T} = TM \oplus T^* M, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)\) with Ševera class \([H]\) with \(H \in \Omega^3_{c1}(M)\) consists an anchor \(\rho : \mathcal{T} \longrightarrow TM\) being the projection, a canonical pairing \(\langle \cdot, \cdot \rangle\) given by

\[
\langle X + \xi, Y + \eta \rangle = \frac{1}{2} (\xi(Y) + \eta(X)), \quad \forall \; X, Y \in \mathfrak{X}(M), \; \xi, \eta \in \Omega^1(M),
\]

an antisymmetric bracket \([\cdot, \cdot]\) given by

\[
[X + \xi, Y + \eta] \triangleq [X, Y] + L_X \eta - L_Y \xi + \frac{1}{2} d(\xi(Y) - \eta(X)) + i_X \wedge \gamma H.
\]

However \([\cdot, \cdot]\) is not a Lie bracket,

\[
[[[e_1, e_2], e_3] + [[[e_2, e_3], e_1] + [[[e_3, e_1], e_2] = dT(e_1, e_2, e_3), \quad \forall \; e_1, e_2, e_3 \in \Gamma(T_{c1}(M)),
\]

\[
T(e_1, e_2, e_3) = \frac{1}{3} (\langle [e_1, e_2], e_3 \rangle + \langle [e_2, e_3], e_1 \rangle + \langle [e_3, e_1], e_2 \rangle).
\]
Courant algebroids \( \subset \) Lie 2-algebroids [SZ, Sect.3,4]

\[
[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] = dT(e_1, e_2, e_3), \quad \forall e_1, e_2, e_3 \in \Gamma(T)
\]

(17)

where \( T(e_1, e_2, e_3) \) is given by

\[
T(e_1, e_2, e_3) = \frac{1}{3}([[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2])
\]

(18)

These two equations show that there is a Lie 2-algebra structure on sections [RW98].
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Courant algebroids $\subset$ Lie 2-algebroids [SZ, Sect.3,4]


$$
\begin{align*}
\rho(X + \xi) &= X, \quad l_1(\alpha) = \alpha, \quad l_2(X + \xi, \alpha) = \nabla_X(\alpha), \\
l_2(X + \xi, Y + \eta) &= [X, Y] + \nabla_X(\eta) - \nabla_Y(\xi), \\
l_3(X + \xi, Y + \eta, Z + \gamma) &= -\omega(X, Y)(\gamma) - \omega(Y, Z)(\xi) - \omega(Z, X)(\eta),
\end{align*}
$$

for all $X, Y, Z \in \Gamma(TM)$, $\xi, \eta, \gamma, \alpha \in \Gamma(T^*M)$. Here $\nabla$ is a $TM$-connection $\nabla$ on $T^*M$, and a 2-form $\omega \in \Omega^2(TM, \text{End}(T^*M, T^*M))$ satisfying

$$
\nabla_{[X, Y]} = [\nabla_X, \nabla_Y] = \omega(X, Y), \quad \forall X, Y \in \Gamma(TM).
$$

This is the case when $H = 0$. 
Courant algebroids $\subset$ Lie 2-algebroids \([SZ, \text{Sect.3,4}]\)


\[
\begin{align*}
\rho(X + \xi) &= X, \quad l_1(\alpha) = \alpha, \quad l_2(X + \xi, \alpha) = \nabla_X(\alpha), \\
l_2(X + \xi, Y + \eta) &= [X, Y] + \nabla_X(\eta) - \nabla_Y(\xi) + c_2(X, Y), \\
l_3(X + \xi, Y + \eta, Z + \gamma) &= -\omega(X, Y)(\gamma) - \omega(Y, Z)(\xi) - \omega(Z, X)(\eta) + c_3(X, Y, Z),
\end{align*}
\]

for all $X, Y, Z \in \Gamma(TM), \, \xi, \eta, \gamma, \alpha \in \Gamma(T^*M)$. Here $\nabla$ is a $TM$-connection $\nabla$ on $T^*M$, and a 2-form $\omega \in \Omega^2(TM, \text{End}(T^*M, T^*M))$ satisfying

\[
\nabla_{[X,Y]} - [\nabla_X, \nabla_Y] = \omega(X, Y), \quad \forall X, Y \in \Gamma(TM).
\]

In general, we must add additional terms

\[
\begin{align*}
c_2(X, Y) &= i_X \wedge Y H, \\
c_3(X, Y, Z) &= d_{\nabla} c_2(X, Y, Z) = \nabla_X c_2(Y, Z) - c_2([X, Y], Z) + c.p.(X, Y, Z).
\end{align*}
\]
We expect that $\mathcal{T}$ serves as a differentiation functor and there should be an integration functor $\mathcal{I}$ which is adjoint to it:

$$
\mathcal{T}: \text{Lie } n\text{-groupoids} \leftrightarrow \text{Lie } n\text{-algebroids}: \mathcal{I}
$$
Differentiation/Integration

We expect that $\mathcal{T}$ serves as a differentiation functor and there should be an integration functor $\mathcal{I}$ which is adjoint to it:

$$
\mathcal{T}: \text{Lie } n\text{-groupoids} \leftrightarrow \text{Lie } n\text{-algebroids}: \mathcal{I}
$$

$\mathcal{T}$:

- A formalization of the notion Lie differentiation in higher geometry has been given by Lurie (deformation context), worked out in diff. Geom. setting by Joost Nuiten, but the result is abstract.
- Ševera [Šev] sketches a functor $\mathcal{T}$ explicitly with idea (inspired by Konsevich 2006).
- It is then being tried to work out (local result) in Theorem 8.28 of, Du Li’s thesis [Li14] (arXiv:1512.04209), but it contains a mistake in Lemma 8.34 (2015).
- In work in progress [?], we complete this result in its global version.
Differentiation/Integration

We expect that $\mathcal{T}$ serves as a differentiation functor and there should be an integration functor $\mathcal{I}$ which is adjoint to it:

$$\mathcal{T}: \text{Lie } n\text{-groupoids} \leftrightarrow \text{Lie } n\text{-algebroids}: \mathcal{I}$$

$\mathcal{I}$:

- Getzler [Get09] integrates nilpotent $L_\infty$-algebras;
- Henriques [Hen08] integrates $L_\infty$-algebras in general with an infinite dimensional model;
- Ševera and Širaň [Šv20] integrates $L_\infty$-algebroid locally;
- work in progress of Wolfson and Rogers integrates $L_\infty$-algebras in a finite dimensional model.
Differential forms on simplicial manifolds

Differential forms on a simplicial manifold $X_\bullet$ live in the de Rham simplicial double complex $(\Omega^\bullet(X_\bullet), d, \delta)$, see [BSS76, Dup76]. Here $d$ is the usual de Rham differential $d : \Omega^q(X_p) \to \Omega^{q+1}(X_p)$, and the $\delta$ is the simplicial differential

$$\delta : \Omega^q(X_{p-1}) \to \Omega^q(X_p), \quad \delta = \sum_{i=0}^{p} (-1)^i d_i^*.$$ 

The associated differential on the total complex is $D = \delta + (-1)^p d$. A special important sub-complex is the so called normalised double complex

$$\hat{\Omega}^\bullet(X_\bullet) = \{ \alpha \in \Omega^\bullet(X_\bullet) \mid s_i^* \alpha = 0, \text{ for all possible } i \text{'s} \},$$

in other words, a differential form $\alpha$ is called normalised if it vanishes on degeneracies. An $m$-shifted $k$-form $\alpha_\bullet$ is

$$\alpha_\bullet = \sum_{i=0}^{m} \alpha_i \text{ with } \alpha_i \in \hat{\Omega}^{k+m-i}(X_i). \quad (19)$$
We say that $\alpha_\bullet$ is **closed** if $D\alpha_\bullet = 0$. We notice that the cut of the range of the form in (19) above is initiated in [PTVV13] to interpret closedness, and nicely explained also in [Get] from the angle of resolution. However, it is different from the “closedness” we define here by $D\alpha_\bullet = 0$. 

Chenchang Zhu (Mathematisches Institut, Göttingen)
Following [Get], we express the non-degeneracy of a $m$-shifted 2-form $\alpha_\bullet$ on a (local) Lie $n$-groupoid $X_\bullet$ in terms of the tangent complex $(\mathcal{T}_\bullet X, \partial)$. 
Following [Get], we express the non-degeneracy of a $m$-shifted 2-form $\alpha_\bullet$ on a (local) Lie $n$-groupoid $X_\bullet$ in terms of the tangent complex $(\mathcal{T}_\bullet X, \partial)$. For a Lie $n$-groupoid $X_\bullet$ and a closed $m$-shifted 2-form $\alpha_\bullet$, we introduce its **IM-form** (infinitesimal-form) $\lambda^{\alpha_\bullet}$ in the following way: for $k \in \mathbb{Z}$ and $v \in \mathcal{T}_k K \subset T_x K_k$, $w \in \mathcal{T}_{m-k} K \subset T_x K_{m-k}$ in the tangent complex at $x \in K_0$, we define

$$
\lambda^{\alpha_\bullet}_x(v, w) := \sum_{\sigma \in Sf_{k, m-k}} (-1)^{\sigma} \alpha_m(T(s_{\sigma(m-1)} \cdots s_{\sigma(k)})v, T(s_{\sigma(k-1)} \cdots s_{\sigma(0)})w)
$$

where $Sf_{k, m-k}$ is the set of $(k, m-k)$-shuffles, and $(-1)^{\sigma}$ is the permutation sign of $\sigma$.

---

³This nice explicit formula of IM form comes from [Get]. Nevertheless, as also indicated in [BCWZ04, BD19] in lower cases, $\lambda^{\alpha_\bullet}$ is probably just the leading term of the full IM-form and does not contain all the infinitesimal information.
It is clear from the definition that \( \lambda^{\alpha\bullet} \) is graded anti-symmetric. Furthermore, \( \lambda^{\alpha\bullet} \) is infinitesimal multiplicative\(^4\), that is for \( u \in \mathcal{T}_{k+1}X \), \( w \in \mathcal{T}_{m-k}X \)

\[
\lambda^{\alpha\bullet}(\partial u, w) + (-1)^{k+1} \lambda^{\alpha\bullet}(u, \partial w) = 0.
\]

(21)

This implies that \( \lambda^{\alpha\bullet}(\cdot, \cdot) \) descends to the homology groups \( H_\bullet(\mathcal{T}X) \).

\(^4\)This together with several other useful statements is stated in [Get] without a proof. Proof can be found in [CZ, App.E], thanks to the master thesis of Florian Dorsch under the guide of the second author.
Non-degeneracy

It is clear from the definition that $\lambda^\alpha\cdot$ is graded anti-symmetric. Furthermore, $\lambda^\alpha\cdot$ is **infinitesimal multiplicative**\(^4\), that is for $u \in \mathcal{T}_{k+1}X$, $w \in \mathcal{T}_{m-k}X$

$$
\lambda^\alpha\cdot(\partial u, w) + (-1)^{k+1} \lambda^\alpha\cdot(u, \partial w) = 0.
$$

(21)

This implies that $\lambda^\alpha\cdot(\cdot, \cdot)$ descends to the homology groups $H_\bullet(\mathcal{T}X)$. Thus we have an induced pairing

$$
\lambda^\alpha\cdot(\cdot, \cdot) : H_k(\mathcal{T}X) \times H_{m-k}(\mathcal{T}X) \to \mathbb{R} \chi_0
$$

(22)

The \textit{m-shifted} 2-form $\alpha\cdot$ is called **non-degenerate** if (??) defines a pointwise non-degenerate pairing.

---

\(^4\)This together with several other useful statements is stated in [Get] without a proof. Proof can be found in [CZ, App.E], thanks to the master thesis of Florian Dorsch under the guide of the second author.
Cotangent complex

A way to reformulate the non-degeneracy is by introducing the cotangent complex \((\mathcal{T}_i^* \mathcal{X}, \partial^*)\) with

\[
\mathcal{T}_i^* \mathcal{X} = (\mathcal{T}_{-i} \mathcal{X})^* \quad \text{and} \quad \partial^* = \partial^t.
\]

Then \(\lambda^{\alpha \bullet}\) being non-degenerate can be rephrased by saying that the morphism

\[
\lambda^{\alpha \bullet, \#} : (\mathcal{T}_\bullet \mathcal{X}, \partial) \to (\mathcal{T}_\bullet^* \mathcal{X}[-m], \partial^t)
\]

\[
\mathbf{v} \mapsto \lambda^{\alpha \bullet, \#}(\mathbf{v}) = \lambda^{\alpha \bullet}(\mathbf{v}, \cdot)
\]

is a quasi-isomorphism of complexes.
Cotangent complex

A way to reformulate the non-degeneracy is by introducing the **cotangent complex** $(\mathcal{T}_i^*X, \partial^*)$ with

$$\mathcal{T}_i^*X = (\mathcal{T}_{-i}X)^*$$

and

$$\partial^* = \partial^t.$$

Then $\lambda^\alpha\cdot$ being non-degenerate can be rephrased by saying that the morphism

$$\lambda^\alpha\cdot,\# : (\mathcal{T}_iX, \partial) \to (\mathcal{T}_i^*X[-m], \partial^t)$$

$$\nu \to \lambda^\alpha\cdot,\#(\nu) = \lambda^\alpha\cdot(\nu, \cdot)$$

is a quasi-isomorphism of complexes.

$$H_\bullet(\mathcal{T}) : 0, 1, \ldots, n$$

$$H^\bullet(\mathcal{T}^*)[-m] : \ldots, m - n, m - n + 1, \ldots, m$$

If $H_0(\mathcal{T}) \neq 0$ and $H_n(\mathcal{T}) \neq 0$, we only have $n$-shifted symplectic forms. If $H_0(\mathcal{T}) = 0$, we could allow $m > n$; if $H_n(\mathcal{T}) = 0$, we could allow $m < n$. 
The pair \((X\bullet, \alpha\bullet)\) is an \(m\)-shifted symplectic Lie \(n\)-groupoid if \(X\bullet\) is a Lie \(n\)-groupoid and \(\alpha\bullet\) is a closed, normalized and non-degenerate \(m\)-shifted 2-form on \(X\bullet\).
Shifted Symplectic Structure

**Definition ([CZ])**

The pair \((X_\bullet, \alpha_\bullet)\) is an \(m\)-shifted symplectic Lie \(n\)-groupoid if \(X_\bullet\) is a Lie \(n\)-groupoid and \(\alpha_\bullet\) is a closed, normalized and non-degenerate \(m\)-shifted 2-form on \(X_\bullet\).

**Example (m=0, n=0)**

A Lie 0-groupoid is the same as a manifold \(M\), with \(M_\bullet\) given by \(M_i = M\) with faces and degeneracies all equal to the identity. The tangent complex is given by \(T_0M = TM\). Recall \(\lambda^{\alpha_\bullet}(\cdot, \cdot) : H_k(TX) \times H_{m-k}(TX) \to \mathbb{R}X_0\). Hence \(M_\bullet\) only admits 0-shifted symplectic structures. \(\omega_\bullet = \omega_0 \in \Omega^2(M)\) is a 0-shifted symplectic structures if and only if it is symplectic. Thus we obtain the following correspondence

\[
\{ \text{Symplectic manifolds} \} \leftrightarrow \{ \text{0-Shifted symplectic Lie 0-groupoids} \}.
\]
Example \((m=0, n=1)\)

Take a Lie \((1-)\)groupoid \(X_1 \rightrightarrows X_0\). A 0-shifted 2-form is simply a 2-form \(\omega \in \Omega^2(X_0)\), and \(D\omega = 0\) if and only if \(d\omega = 0\) and \(s^*\omega = t^*\omega\). The normalized condition is empty. The tangent complex is the associated Lie algebroid \(A \to X_0\) (p25), and the non-degeneracy is equivalent to

\[
\ker \rho = 0 \quad \text{and} \quad \omega|_{\text{coker} \rho} \text{ non-degenerate.}
\]

Thus \(A = D \subseteq TM\) defines a regular distribution on \(M\) that carries a non-degenerate 2-form on the transversal directions. In other words, a 0-shifted symplectic Lie 1-groupoid gives a precise mathematical formulation for the sentence: “The leaf space \(F = M/D\), of a regular distribution \(D\) on a manifold \(M\), is symplectic", and this is what we shall expect for a symplectic differentiable stack.
Example (m=1, n=1)

A twisted presymplectic Lie groupoid [BCWZ04], a.k.a. quasi-symplectic Lie groupoid [Xu04], is a triple \((K \Rightarrow M, \omega, H)\) where \(K \Rightarrow M\) is a Lie groupoid, \(\omega \in \Omega^2(K)\) and \(H \in \Omega^3(M)\) satisfying

\[
\delta \omega = 0, \quad d \omega = \delta H, \quad dH = 0 \quad \text{and} \quad \ker \omega_x \cap \ker T_s \cap \ker T_x \Rightarrow t = 0 \quad \forall x \in M.
\]

As in the previous example we fix \(s = d_0, \ t = d_1, \ A = \ker T_s |_M\) and \(\rho = T t |_A\). The equation \(\delta \omega = 0\) implies that \(\omega_x = 0\) for \(x \in M\). Hence \(\omega_\bullet := \omega + H\) is a normalized 1-shifted 2-form. The three equations from the left are equivalent to \(D \omega_\bullet = 0\). The last condition is equivalent to the non-degeneracy condition for \(\omega_\bullet\). Thus

\[
\{ \text{Twisted presymplectic Lie groupoids} \} \Rightarrow \{ \text{1-shifted symplectic Lie 1-groupoids} \}
\]

Thus symplectic groupoid (differentiated to Poisson manifold) is a particular case of a 1-shifted symplectic Lie 1-groupoid.
Example (m=2, n=1)

Let \( G \) be a Lie group with Lie algebra \((\mathfrak{g}, \langle \cdot, \cdot \rangle)\) a quadratic Lie algebra. Then we can define the following differential forms on the nerve \( NG \) (p9):

\[
\Omega = \langle d^*_2 \theta^l, d^*_0 \theta^r \rangle \in \Omega^2(NG_2) \quad \text{and} \quad \Theta = \frac{1}{6} \langle \theta^l, [\theta^l, \theta^l] \rangle \in \Omega^3(NG_1) \quad (23)
\]

where \( \theta^l, \theta^r \in \Omega^1(G; \mathfrak{g}) \) are the left and right Maurer-Cartan 1-forms on \( G \). Then \((NG, \Omega - \Theta)\) is a 2-shifted symplectic Lie 1-group \([\text{Bry93, Wei95, PS20}]\).
**Classifying stack BG**

**Example (m=2, n=2)**

Let $G$ be a Lie group, there is a Lie 2-group $G_{•}$:

$$G_{•} = \cdots \Omega G \Rightarrow P_e G \Rightarrow pt.$$

The face maps $d_i : \Omega G \to P_e G$ for $i = 0, 1, 2$ are defined as

$$d_0(\tau)(t) = \tau\left(\frac{t}{3}\right), \quad d_1(\tau)(t) = \tau\left(1 - \frac{t}{3}\right), \quad d_2(\tau)(t) = \tau\left(\frac{1}{3} + \frac{t}{3}\right) \cdot \tau\left(\frac{1}{3}\right)^{-1},$$

for $t \in [0, 1]$, and Segal’s symplectic 2-form:

$$\omega_\tau(a, b) = \int_{S^1} \langle \hat{a}'(t), \hat{b}(t) \rangle \ dt \in \Omega^2(\Omega G), \quad (24)$$

where $\tau \in \Omega G$, $a, b \in T_\tau \Omega G$, $\hat{a}(t) = L_{\tau(t)^{-1}} a(t)$, $\hat{b}(t) = L_{\tau(t)^{-1}} b(t)$.

Then $(G, \omega)$ is a 2-shifted symplectic Lie 2-group [CZ].
Recall that two Lie $n$-groupoids $X_\bullet$ and $Y_\bullet$ are **Morita equivalent** iff there is a third Lie $n$-groupoid $Z_\bullet$ with hypercovers $f_\bullet : Z_\bullet \to X_\bullet$ and $g_\bullet : Z_\bullet \to Y_\bullet$. 
Recall that two Lie $n$-groupoids $X_\bullet$ and $Y_\bullet$ are **Morita equivalent** iff there is a third Lie $n$-groupoid $Z_\bullet$ with hypercovers $f_\bullet : Z_\bullet \to X_\bullet$ and $g_\bullet : Z_\bullet \to Y_\bullet$.

**Definition**

Let $(X_\bullet, \alpha_\bullet)$ and $(Y_\bullet, \beta_\bullet)$ be two $m$-shifted symplectic Lie $n$-groupoids. We say that $(X_\bullet, \alpha_\bullet)$ and $(Y_\bullet, \beta_\bullet)$ are **symplectic Morita equivalent** if there exists $(Z_\bullet, \phi_\bullet)$ another Lie $n$-groupoid with $\phi_\bullet$ an $(m-1)$-shifted 2-form and hypercovers $f_\bullet$, $g_\bullet$ satisfying

$$X_\bullet \xleftarrow{f_\bullet} Z_\bullet \xrightarrow{g_\bullet} Y_\bullet,$$

and

$$f_\bullet^* \alpha_\bullet - g_\bullet^* \beta_\bullet = D\phi_\bullet.$$  

(25)
Recall that two Lie \( n \)-groupoids \( X \) and \( Y \) are Morita equivalent iff there is a third Lie \( n \)-groupoid \( Z \) with hypercovers \( f : Z \rightarrow X \) and \( g : Z \rightarrow Y \).

**Definition**

Let \((X, \alpha)\) and \((Y, \beta)\) be two \( m \)-shifted symplectic Lie \( n \)-groupoids. We say that \((X, \alpha)\) and \((Y, \beta)\) are symplectic Morita equivalent if there exists \((Z, \phi)\) another Lie \( n \)-groupoid with \( \phi \) an \((m-1)\)-shifted 2-form and hypercovers \( f, g \) satisfying

\[
X \xleftarrow{f} Z \xrightarrow{g} Y, \quad \text{and} \quad f^* \alpha - g^* \beta = D \phi.
\]  

This extra condition for the match of the symplectic form means that the map \( f \times g : Z \rightarrow X \times Y \) is Lagrangian\(^5\), as defined in [Saf16], with respect to the shifted symplectic structure \( \alpha - \beta \). This Morita equivalence coincides with the one defined in [Xu04] when \( m = n = 1 \) via bibundles.

\(^5\)Here we appreciate the private communication with Pavel Safronov.
**Example (Strict morphisms)**

If $f_\bullet : Y_\bullet \to X_\bullet$ is a hypercover of Lie $n$-groupoids, and $\alpha_\bullet$ is an $m$-shifted symplectic form on $X_\bullet$, then one can show $(Y_\bullet, f^* \alpha_\bullet)$ is also an $m$-shifted symplectic Lie $n$-groupoid. Moreover $(X_\bullet, \alpha_\bullet)$ and $(Y_\bullet, f^* \alpha_\bullet)$ are symplectic Morita equivalent via $(Y_\bullet, 0)$ with one leg $f_\bullet$ and the other leg the identity morphism,

$$(Y_\bullet, f^* \alpha_\bullet) \leftrightarrow (Y_\bullet, 0) \xrightarrow{\text{id}_\bullet} (X_\bullet, \alpha_\bullet).$$
Examples

Example (Strict morphisms)

If $f : Y \rightarrow X$ is a hypercover of Lie $n$-groupoids, and $\alpha$ is an $m$-shifted symplectic form on $X$, then one can show $(Y, f^* \alpha)$ is also an $m$-shifted symplectic Lie $n$-groupoid. Moreover $(X, \alpha)$ and $(Y, f^* \alpha)$ are symplectic Morita equivalent via $(Y, 0)$ with one leg $f$ and the other leg the identity morphism,

$$(Y, f^* \alpha) \xleftarrow{id} (Y, 0) \xrightarrow{f} (X, \alpha).$$

Let $(X, \alpha)$ be an $m$-shifted symplectic Lie $n$-groupoid and $\phi$ an $(m-1)$-shifted 2-form. Then $(X, \alpha + D\phi)$ is again an $m$-shifted symplectic Lie $n$-groupoid and

$$(X, \alpha + D\phi) \xleftarrow{id} (X, \phi) \xrightarrow{id} (X, \alpha).$$

is a symplectic Morita equivalence. We call this sort of symplectic Morita equivalence a gauge transformation.
Comparing two models of classifying stack $BG$

We have two models $(NG, (\Omega, \Theta))$ and $(G = \ldots \Omega G \xrightarrow{\equiv} P_e G \xrightarrow{\equiv} pt, \omega)$ for the 2-shifted symplectic structure on $BG$. 
We have two models \((NG, (\Omega, \Theta))\) and \((G = \ldots \Omega G \rightleftharpoons P_e G \rightleftharpoons pt, \omega)\) for the 2-shifted symplectic structure on \(BG\). The evaluation map \(ev_1 : P_e G \rightarrow G\) defined by evaluating at time 1, that is \(ev_1(\gamma) = \gamma(1)\), extends to a simplicial morphism

\[
ev : G_\bullet \rightarrow NG_\bullet,
\]

with \(ev_2 : \Omega G \rightarrow G \times G\), and \(ev_3 : G_3 \rightarrow G \times^3\) given by

\[
ev_2(\tau) = (\tau(\frac{2}{3})\tau(\frac{1}{3})^{-1}, \tau(\frac{1}{3})) , \quad ev_3(\tau_0, \tau_1, \tau_2) = (\tau_2(\frac{2}{3})\tau_2(\frac{1}{3})^{-1}, \tau_0(\frac{2}{3})\tau_0(\frac{1}{3})^{-1}, \tau_0(\frac{2}{3})\tau_0(\frac{1}{3})^{-1}, \tau_0(\frac{2}{3})\tau_0(\frac{1}{3})^{-1})
\]
Comparing two models of classifying stack $BG$

We have two models $(NG, (\Omega, \Theta))$ and $(G = \ldots \Omega G \rightrightarrows P_G \rightrightarrows \text{pt}, \omega)$ for the 2-shifted symplectic structure on $BG$. The evaluation map $ev_1 : P_G \to G$ defined by evaluating at time 1, that is $ev_1(\gamma) = \gamma(1)$, extends to a simplicial morphism

$$ev_\bullet : G_\bullet \to NG_\bullet,$$

with $ev_2 : \Omega G \to G \times G$, and $ev_3 : G_3 \to G^{\times 3}$ given by

$$ev_2(\tau) = (\tau(\frac{2}{3})\tau(\frac{1}{3})^{-1}, \tau(\frac{1}{3})), \quad ev_3(\tau_0, \tau_1, \tau_2) = (\tau_2(\frac{2}{3})\tau_2(\frac{1}{3})^{-1}, \tau_0(\frac{2}{3})\tau_0(\frac{1}{3}))$$

Lemma

Let $G$ be a connected and simply connected Lie group. Then the evaluation map $ev_\bullet : G_\bullet \to NG_\bullet$ is an hypercover.
On the infinitesimal level, we notice here that the evaluation map
\[ \text{ev}_\bullet : \mathcal{G}_\bullet \to N\mathcal{G}_\bullet \]
duces a map between the Lie 2-algebras
\[ \tilde{\partial} = (\Omega \mathfrak{g} \xrightarrow{\partial} P_0 \mathfrak{g}) \text{ and } \mathfrak{g} = (0 \to \mathfrak{g}) \]
given by
\[
\text{Lie}(\text{ev})_\bullet : \tilde{\partial} \to \mathfrak{g}, \quad \text{Lie}(\text{ev})_1(u) = u(1) \text{ and } \text{Lie}(\text{ev})_2(a) = a(1) = 0,
\]
for \( u \in \Omega \mathfrak{g} \) and \( a \in P_0 \mathfrak{g} \). One can check it is a quasi-isomorphism of Lie 2-algebras.
Let $G$ be a connected and simply connected Lie group with quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. The 2-shifted symplectic Lie 2-groups $(G\bullet, \omega\bullet)$ and $(NG\bullet, \frac{1}{2}\Omega\bullet)$ are symplectic Morita equivalent via

$$(G\bullet, \omega\bullet) \xleftarrow{\text{Id} \bullet} (G\bullet, \omega_P\bullet) \xrightarrow{\text{ev} \bullet} (NG\bullet, \frac{1}{2}\Omega\bullet)$$

where $\omega_P\bullet = \omega_P + 0$ is a 1-shifted form with

$$\omega_P^\gamma(u, v) = \frac{1}{2} \int_0^1 \langle \hat{u}'(t), \hat{v}(t) \rangle - \langle \hat{u}(t), \hat{v}'(t) \rangle dt \in \Omega^2(G_1) \in \Omega^2(P_e G) .$$
Symplectic Morita Equivalence

**Theorem**

Let $G$ be a connected and simply connected Lie group with quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. The 2-shifted symplectic Lie 2-groups $(\mathcal{G}_\bullet, \omega_\bullet)$ and $(NG_\bullet, \frac{1}{2} \Omega_\bullet)$ are symplectic Morita equivalent via

$$(\mathcal{G}_\bullet, \omega_\bullet) \xleftarrow{\text{Id}_\bullet} (\mathcal{G}_\bullet, \omega_P^\bullet) \xrightarrow{\text{ev}_\bullet} (NG_\bullet, \frac{1}{2} \Omega_\bullet)$$

where $\omega_P^\bullet = \omega^P + 0$ is a 1-shifted form with

$$\omega_P^\gamma(u, v) = \frac{1}{2} \int_0^1 \langle \hat{u}'(t), \hat{v}(t) \rangle - \langle \hat{u}(t), \hat{v}'(t) \rangle dt \in \Omega^2(\mathcal{G}_1) \in \Omega^2(P_e G).$$

$$1/2(\text{ev}_1^* \Theta, \text{ev}_2^* \Omega) - \omega = D\omega^P = (-d\omega^P, \delta\omega^P),$$

where $\omega^P|_{\Omega G} = \omega$. 
Theorem

Let $G$ be a connected and simply connected Lie group with quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. The 2-shifted symplectic Lie 2-groups $(G\bullet, \omega\bullet)$ and $(NG\bullet, \frac{1}{2}\Omega\bullet)$ are symplectic Morita equivalent via

$$(G\bullet, \omega\bullet) \xleftarrow{\text{Id}\bullet} (G\bullet, \omega^P\bullet) \xrightarrow{\text{ev}\bullet} (NG\bullet, \frac{1}{2}\Omega\bullet)$$

where $\omega^P = \omega^P + 0$ is a 1-shifted form with

$$\omega^P(\gamma, \nu) = \frac{1}{2} \int_0^1 \langle \hat{u}'(t), \hat{v}(t) \rangle - \langle \hat{u}(t), \hat{v}'(t) \rangle \, dt \in \Omega^2(G_1) \in \Omega^2(\text{Pe} G).$$

$$1/2(ev_1^*\Theta, ev_2^*\Omega) - \omega = D\omega^P = (-d\omega^P, \delta\omega^P), \text{ where } \omega^P|_{\Omega G} = \omega.$$

- **transgression map** $\mathbb{T} : \Omega^k(M) \to \Omega^{k-1}(\text{PM})$, $\beta \mapsto \int_0^1 ev^* \beta$,
- $ev_1^*\Theta = d\mathbb{T}(\Theta)$, $\mathbb{T}(\Theta) = d\alpha^P - 2\omega^P$, with

$$\alpha^P(\gamma, u) = \int_0^1 \langle \gamma^{-1}\gamma', \gamma^{-1}u' \rangle \, dt \in \Omega^1(G_1)$$
Double Lie groups $\mathbb{G}_{\bullet,\bullet}, G_{\bullet,\bullet}$ given by the square

\[
\begin{array}{ccc}
\Omega G & \to & pt \\
\downarrow & & \downarrow \\
P_e G & \to & pt \\
\end{array}
\quad \quad
\begin{array}{ccc}
G & \to & pt \\
\downarrow & & \downarrow \\
G & \to & pt \\
\end{array}
\]
Double Lie groups $G_{\bullet,\bullet}, G_{\bullet,\bullet}$ given by the square

$$
\begin{align*}
\Omega G & \xrightarrow{\text{ev}_{\bullet,\bullet}} pt, \\
PeG & \xrightarrow{} pt
\end{align*}
\quad \text{and} \quad
\begin{align*}
G & \xrightarrow{} pt, \\
G & \xrightarrow{} pt
\end{align*}
$$

(26)
Double Lie groups $G_{\bullet, \bullet}$, $G_{\bullet, \bullet}$ given by the square

$$
\begin{array}{c}
\begin{array}{c}
\Omega G \longrightarrow pt \\
\downarrow \downarrow \\
P_e G \longrightarrow pt
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
ev_{\bullet, \bullet} \quad G \longrightarrow pt \\
\downarrow \\
G \longrightarrow pt
\end{array}
\end{array}
$$

equipped with closed shifted forms

$$
\omega_{\bullet, \bullet} = \begin{pmatrix}
-\eta & \omega & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \Omega_{\bullet, \bullet} = \begin{pmatrix}
0 & 0 & 0 \\
\Omega & -\Theta & 0
\end{pmatrix},
$$

$$
\eta(\tau_1, \tau_2)((a_1, a_2)) = \int_0^1 \langle \tau_2 \tau_2^{-1}, \tau_1^{-1} a_1' \rangle dt \in \Omega^1(\Omega G \times 2) = \Omega^1(\mathbb{G}_{1,2})
$$
Double groupoid models (convenient for dynamic system)

Double Lie groups $G_{\bullet,\bullet}$, $G_{\bullet,\bullet}$ given by the square

$$
\begin{align*}
\Omega G & \xrightarrow{\ev_{\bullet,\bullet}} pt, & G & \xrightarrow{\ev_{\bullet,\bullet}} pt, \\
P_e G & \xrightarrow{-} pt & G & \xrightarrow{-} pt
\end{align*}
$$

(26)

equipped with closed shifted forms

$$
\omega_{\bullet,\bullet} = \begin{pmatrix}
-\eta & \omega & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \Omega_{\bullet,\bullet} = \begin{pmatrix}
0 & 0 & 0 \\
\Omega & -\Theta & 0
\end{pmatrix},
$$

$$
\eta(\tau_1,\tau_2)((a_1, a_2)) = \int_0^1 \langle \tau_2^{-1}, \tau_1^{-1} a'_1 \rangle dt \in \Omega^1(\Omega G^\times 2) = \Omega^1(G_{1,2}) \quad (27)
$$
satisfies,

$$
-\omega_{\bullet,\bullet} - \frac{1}{2} \ev_{\bullet,\bullet}^* \Omega_{\bullet,\bullet} = D\left(\frac{1}{2} \alpha_{\bullet,\bullet}\right).
$$
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