Obstructions to parabolic quadratic rational maps

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Quadratic rational maps

We consider quadratic rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
A quadratic rational map has:

- two distinct critical points, c_1 and c_2,
- three fixed points, with multipliers μ, λ and γ respectively.
- a Julia set J(f), which is either
  - connected, or
  - a Cantor set $\Leftrightarrow$ both critical points are in the immediate basin of a fixed point with multiplier $\lambda \in \mathbb{D} \cup \{1\}$. (Milnor)

Denote by Rat_2 the space of quadratic rational maps

Moduli space of quadratic rational maps

...and by $\mathcal{M}_2$ its moduli space, i.e the quotient modulo Möbius conjugacy

$$\mathcal{M}_2 = Rat_2/Rat_1 \cong \mathbb{C}^2 \quad \text{(Milnor)}.$$

Following Milnor we consider loci in $\mathcal{M}_2$,

$$Per_1(\lambda) = \{[f] \in \mathcal{M}_2 | \ f \text{ has a fixed point of multiplier } \lambda\}.$$

For each $\lambda \in \mathbb{C}$, $Per_1(\lambda) \cong \mathbb{C}$ with $\sigma = \mu\gamma : Per_1(\lambda) \to \mathbb{C}$ a natural isomorphism. (Milnor)

$Per_1(0)$ is parametrized by the family $Q_c(z) = z^2 + c$, where $\sigma = 4c$.

Parabolic lines $Per_1(\omega)$

We focus on parabolic lines $Per_1(\omega)$, i.e. $\omega = e^{2\pi i p/q}$, $p/q \neq 0/1$.

- Here all Julia sets are connected, so we need another dichotomy in these cases...
- Maps f_σ in $Per_1(\omega)$ have one free critical point (but both critical points are *active* in $\mathcal{M}_2$ in the sense of McMullen).
- We say that c_1 and c_2 are *Fatou related* if they are in the same grand orbit of Fatou components.
- We say that c_1 and c_2 are *related* if

$$\omega(c_1) = \{z_0\} = \omega(c_2),$$

where z_0 is the (persistent) parabolic fixed point and $\omega(c_i)$ is the ω–limit set for the critical point c_i for the map f_σ.

The Fatou relatedness locus

For $\omega = e^{2\pi i p/q}$ $(\lambda \in \mathbb{C})$, we define the *Fatou relatedness locus*:

$$\mathcal{R}^{\omega} = \{[f] \in Per_1(\omega) : c_1, c_2 \text{ are Fatou related}\}, \qquad (1)$$

its complement

$$M^{\omega} = Per_1(\omega) \backslash \mathcal{R}^{\omega} = \{[f] \in Per_1(\omega) : c_1, c_2 \text{ are not Fatou related}\}.$$

and an extended relatedness locus

$$\hat{\mathcal{R}}^{\omega} = \{[f] \in Per_1(\omega) : \omega(c_1) = \{z_0\} = \omega(c_2)\}. \qquad (2)$$

The parabolic line $Per_1(e^{2\pi i/3})$

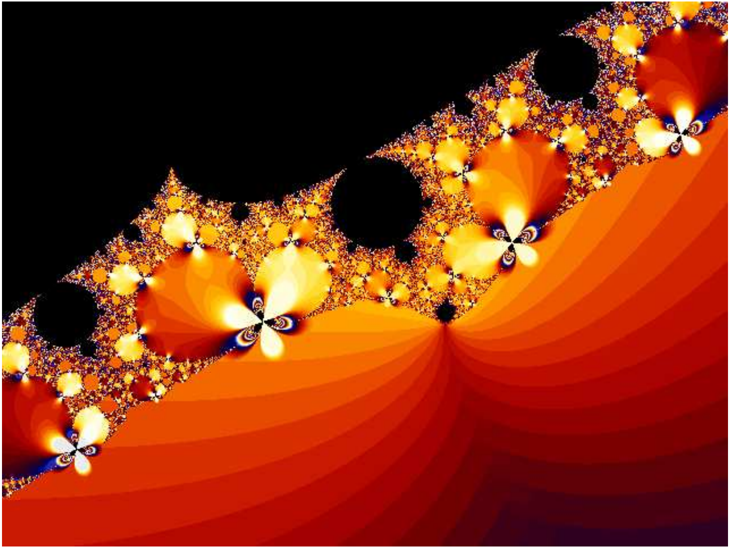


Figure: $Per_1(e^{2\pi i/3})$.

The parabolic line $Per_1(e^{2\pi i/5})$

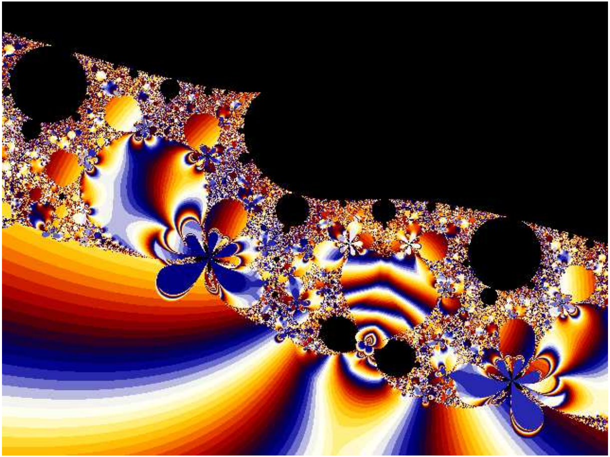


Figure: $Per_1(e^{2\pi i/5})$.

An old friend – the fat Douady rabbit

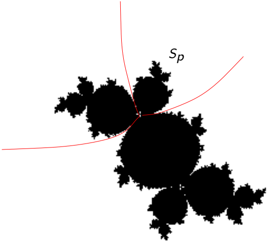
Consider the polynomial $P_\omega(z) = \omega z + z^2$, with critical point $-\omega/2$, let as usual K_ω denote its filled–in Julia set.

$$\text{Define} \quad \hat{K}_\omega = \{z \in \hat{\mathbb{C}} : \omega(z) = \{0\}\}.$$

. . .

The parabolic fixed point 0 is the landing point of q external rays, performing a p/q rotation around 0.

The critical value sector is denoted S_p.



A conjecture

It is a standing conjecture that $Per_1(\omega) = Per_1(e^{2\pi i p/q})$ should be understood as the mating of $M \setminus L_{-p/q}$ with $K_\omega \setminus S_p$, the filled–in Julia set K_ω of the fat p/q–rabbit, without the critical value sector S_p.

We present here a model for $\hat{\mathcal{R}}^\omega \subset Per_1(\omega)$ (inspired by Douady & Hubbard, Goldberg & Keen, Milnor, Tan Lei & Petersen) which is slightly different from what the conjecture suggests it should be.

In this talk: we shall focus on why everything in the sector S_p has to be left out, or in other words why this sector corresponds to obstructed parameters.

Our model suggests that $Per_1(\omega)$ should be described as a "doctored" mating.

A model for $\hat{\mathcal{R}}^\omega$ – a "doctored" fat p/q rabbit

Consider again the polynomial P_ω and let ϕ_ω be the Fatou coordinate normalized so that $\phi_\omega \circ P_\omega = 1/q + \phi_\omega$ and $\phi_\omega(-\omega/2) = 0$.

Let $\bigcup_{j=0}^{q-1} \mathcal{P}^j_{1/q}$ be the q c.c.s of $\phi_\omega^{-1}(\{z = x + iy : x > 1/q\})$ with 0 on their boundaries.

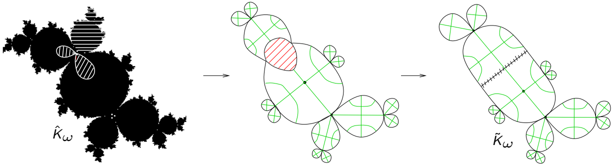
Definition: Two points $z_1 \in \partial\mathcal{P}^j$ and $z_2 \in \partial\mathcal{P}^k$ are called equivalent modulo p/q, written $z_1 \sim_{\frac{p}{q}} z_2$, if the following two conditions are satisfied:

- $j + k = 2p \; mod \; q$, and
- $\phi_\omega(z_1) + \phi_\omega(z_2) = 2/q$.

$**$

A model for $\hat{\mathcal{R}}^\omega$

Definition: Let $\tilde{K}_\omega = \left(\hat{K}_\omega \setminus (S_p \cup \bigcup_{j=0}^{q-1} \mathcal{P}^j_{1/q})\right) /_{\sim_{p/q}}$

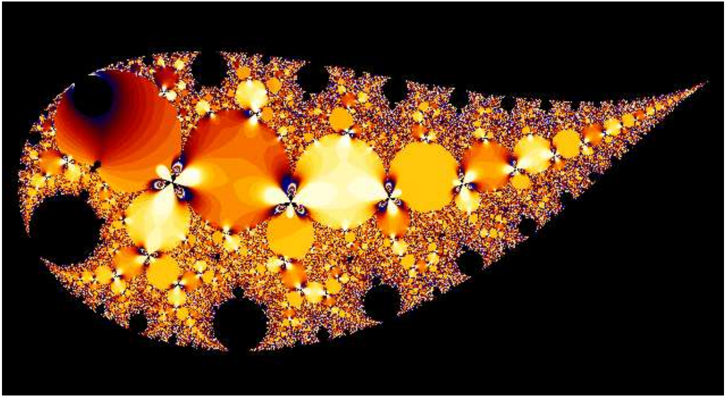


Let $\nu_\omega \subset \tilde{K}_\omega$ denote the scar after this operation. From the Fatou coordinate we then define a *bubble–tree*, denoted $\tilde{\mathcal{T}}^\omega$, in $\tilde{K}_\omega$:

$$\tilde{\mathcal{T}}^\omega = \left(\phi_\omega^{-1}(\mathbb{R}) \cup \nu_\omega \cup \bigcup_{n\geq 0} P_\omega^{-n}(0)\right) \cap \tilde{K}_\omega.$$

Theorem: There exists a bijective map, $\chi^\omega : \hat{\mathcal{R}}^\omega \to \tilde{K}_\omega$, encoding dynamics, which is conformal on the interior of the domain of definition, and so that $(\chi^\omega)^{-1}$ is continuous along the bubble–tree $\tilde{\mathcal{T}}^\omega$.

Proof by picture...



... and the dynamical conjugacy behind it

Consider a map $g_\sigma \in \hat{\mathcal{R}}^\omega$. It has a parabolic fixed point, a basin of attraction Λ^ω and a corresponding q–cycle of components, $B_0, ..., B_{q-1}$, in the immediate basin.
Let $\phi_\sigma : \Lambda^\omega \to \mathbb{C}$ be a Fatou coordinate normalized so that

$$\phi_\sigma^{(j+p) mod q} \circ g_\sigma = \phi_\sigma^j + 1/q$$

Let $\mathcal{P}_r^j \subset B_j$ be an attracting petal, so that $\phi_\sigma|_{\mathcal{P}_r^j}$ is conformal onto the right half plane $\mathbb{H}_r = \{z = x + iy : x > r\}$.
Let R be smallest so that each $\phi_\sigma : \mathcal{P}_R^j \to \mathbb{H}_R$ is conformal.
Then (at least) one petal $\mathcal{P}^j$ contains a critical point in its boundary, called the closest critical point c_1 (with value v_1).
The other critical point is then denoted c_2 (with value v_2).

... and the dynamical conjugacy behind it

Further normalize/name so that $c_1 \in B_0$ and $\phi_\sigma(c_1) = 0$.
Define a dynamical, conformal conjugacy

$$\eta_{\sigma,\omega} = \phi_\omega^{-1} \circ \phi_\sigma : \overline{\bigcup_{j=0}^{q-1} \mathcal{P}_0^{j,\sigma}} \to \overline{\bigcup_{j=0}^{q-1} \mathcal{P}_0^{j,\omega}}$$

between the closures of the attracting flowers. Furthermore, $\eta_{\sigma,\omega}$ extends by iterated lifting to the dynamics, until the second critical value v_2 is in the domain of $\eta_{\sigma,\omega}$.

Then, $\chi^\omega : \hat{\mathcal{R}}^\omega \to \tilde{K}_\omega$ is (essentially) defined by $\chi^\omega(\sigma) = \eta_{\sigma,\omega}(v_2)$.

(... well, for some parameters, f.ex. degenerate parabolic parameters there is more to the story, but that's another story)

Obstructed parameters

Our result on obstructed parameters:

Theorem: For all g_σ in $\hat{\mathcal{R}}^\omega$, $\eta_{\sigma,\omega}(v_2) \in \hat{K}_\omega \setminus S_p$.

Proof: Assume that there exists a g_σ in $\hat{\mathcal{R}}^\omega$, so that $x = \eta_{\sigma,\omega}(v_2) \in \hat{K}_\omega \cap S_p$.

By the Riemann-Hurwitz formula, x is not in B_p. Assume x is pre–fixed or pre–critical of generation $n > 1$. Then g_σ is conjugate to P_ω up to this level.

A hypothetical example with generation 3... on the blackboard

Obstructed parameters

Some notation...

- Let l_p be the curve in $\phi_\sigma^{-1}(\mathbb{R})$ which connects the parabolic fixed point and the critical value v_2. Note that $v_1 \in l_p$.
- Let $\hat{l}_p$ be the truncation of l_p connecting v_1 to v_2.
- Let C be the pre–image of $\hat{l}_p$ under g_σ.
- The closed curve C is mapped 2–to–1 onto $\hat{l}_p$. It intersects the boundary of B_0 at two places, namely at two pre–images of the parabolic fixed point, of same generation $k = 1 + qm$, for some m.
- Let ν be a curve, "following Fatou equipotential–curves" (with the least number of intersections with ∂B_0) connecting these two intersections through the parabolic fixed point.

Obstructed parameters

- Pull back C' $q-1$ times (each time choosing pre–image in C_{ext}), to a simple closed curve l_1 passing through B_p, containing the parabolic fixed point and surrounding $\hat{I}_p$.
- Denote by V_1 the topological disk with l_1 as its boundary.
- By iterated pull–back with g_σ^q, we obtain a sequence of simple closed curves l_n, homotopic with respect to the set of iterated forward images of the critical points, pairwise disjoint except for the parabolic fixed point, and so that $l_n \to l_{n-1}$ 1–to–1 by g_σ^q.
- Denote by V_n the "croissant" bounded by $l_{n-1} \cup l_n \cup \{z_0\}$, let $\Omega = \bigcup_{n\geq 2} V_n \subseteq C_{ext}$ and $Q = \Omega / g_\sigma^q$.

Then $Q \cong \mathbb{C}/\mathbb{Z} \Rightarrow$ a contradiction.

Obstructed parameters

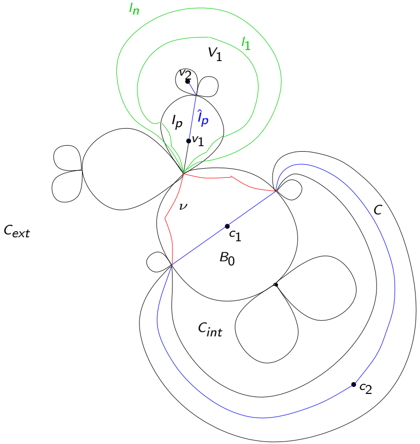


Figure: A forbidden fat rabbit.

A *relatively hyperbolic* Julia set

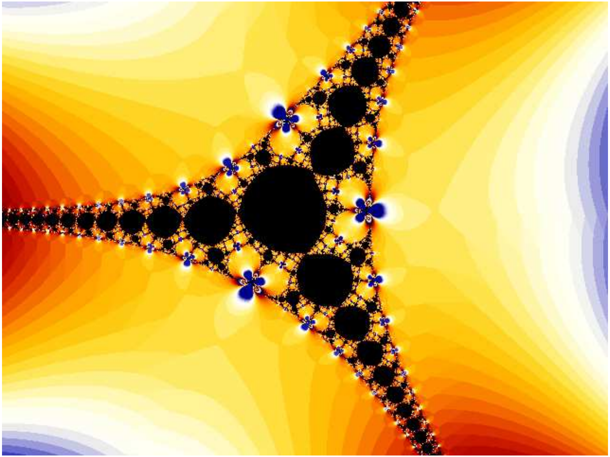


Figure: A basilica and the fat rabbit.

A *parabolic bitransitive* Julia set

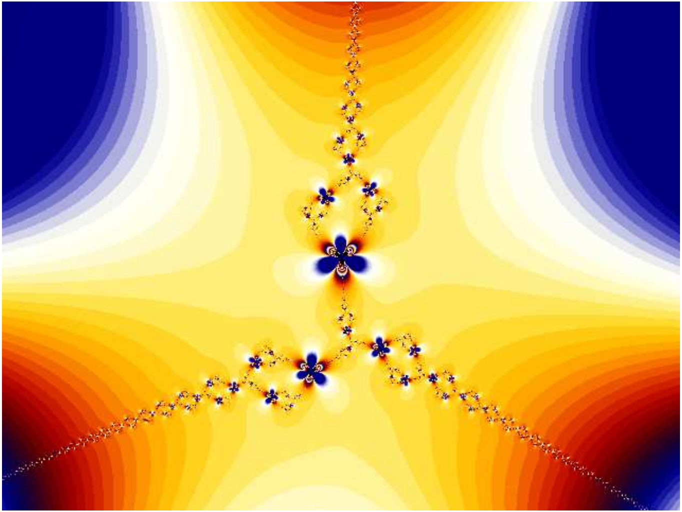


Figure: A “bitransitive fat rabbit”.

A *parabolic capture* Julia set

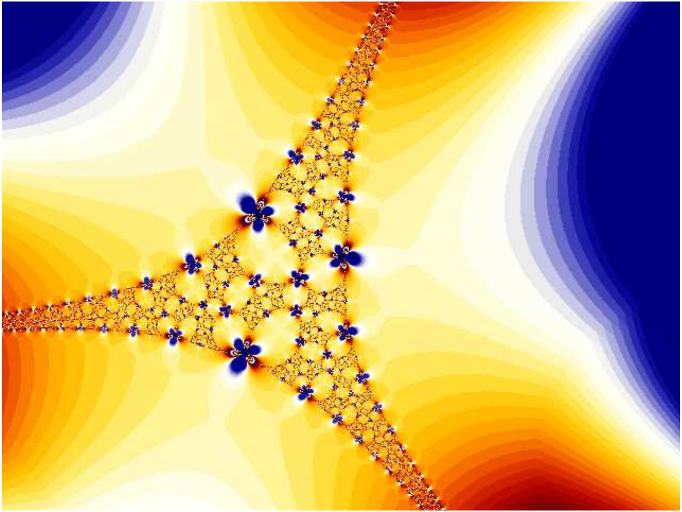


Figure: A "capture fat rabbit".

Thank you for your attention and thanks to the organizers!