Obstructions to parabolic quadratic rational maps

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We consider quadratic rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. A quadratic rational map has:

- two distinct critical points, $c_1$ and $c_2$,
- three fixed points, with multipliers $\mu$, $\lambda$ and $\gamma$ respectively.
- a Julia set $J(f)$, which is either connected, or a Cantor set $\iff$ both critical points are in the immediate basin of a fixed point with multiplier $\lambda \in \mathbb{D} \cup \{1\}$. (Milnor)

Denote by $Rat_2$ the space of quadratic rational maps
...and by $\mathcal{M}_2$ its moduli space, i.e the quotient modulo Möbius conjugacy

$$\mathcal{M}_2 = \text{Rat}_2 / \text{Rat}_1 \cong \mathbb{C}^2 \quad \text{(Milnor)}.$$ 

Following Milnor we consider loci in $\mathcal{M}_2$,

$$\text{Per}_1(\lambda) = \{ [f] \in \mathcal{M}_2 | f \text{ has a fixed point of multiplier } \lambda \}.$$ 

For each $\lambda \in \mathbb{C}$, $\text{Per}_1(\lambda) \cong \mathbb{C}$ with $\sigma = \mu \gamma : \text{Per}_1(\lambda) \to \mathbb{C}$ a natural isomorphism. \text{(Milnor)}

$\text{Per}_1(0)$ is parametrized by the family $Q_c(z) = z^2 + c$, where $\sigma = 4c$. 
We focus on parabolic lines $Per_1(\omega)$, i.e. $\omega = e^{2\pi ip/q}$, $p/q \neq 0/1$.

- Here all Julia sets are connected, so we need another dichotomy in these cases...
- Maps $f_\sigma$ in $Per_1(\omega)$ have one free critical point (but both critical points are active in $M_2$ in the sense of McMullen).
- We say that $c_1$ and $c_2$ are **Fatou related** if they are in the same grand orbit of Fatou components.
- We say that $c_1$ and $c_2$ are **related** if

$$\omega(c_1) = \{z_0\} = \omega(c_2),$$

where $z_0$ is the (persistent) parabolic fixed point and $\omega(c_i)$ is the $\omega$–limit set for the critical point $c_i$ for the map $f_\sigma$. 
For $\omega = e^{2\pi ip/q}$ ($\lambda \in \mathbb{C}$), we define the *Fatou relatedness locus*:

$$\mathcal{R}^\omega = \{ [f] \in Per_1(\omega) : c_1, c_2 \text{ are Fatou related} \},$$

(1)

its complement

$$M^\omega = Per_1(\omega) \setminus \mathcal{R}^\omega = \{ [f] \in Per_1(\omega) : c_1, c_2 \text{ are not Fatou related} \}.$$

and an extended relatedness locus

$$\hat{\mathcal{R}}^\omega = \{ [f] \in Per_1(\omega) : \omega(c_1) = \{z_0\} = \omega(c_2) \}.$$  

(2)
The parabolic line $\text{Per}_1(e^{2\pi i/3})$

**Figure:** $\text{Per}_1(e^{2\pi i/3})$. 
The parabolic line $\text{Per}_1(e^{2\pi i/5})$.

Figure: $\text{Per}_1(e^{2\pi i/5})$. 
Consider the polynomial $P_\omega(z) = \omega z + z^2$, with critical point $-\omega/2$, let as usual $K_\omega$ denote its filled−in Julia set.

Define $\hat{K}_\omega = \{z \in \hat{\mathbb{C}} : \omega(z) = \{0\}\}$.

The parabolic fixed point 0 is the landing point of $q$ external rays, performing a $p/q$ rotation around 0. The critical value sector is denoted $S_p$. 

An old friend – the fat Douady rabbit
A conjecture

It is a standing conjecture that $\text{Per}_1(\omega) = \text{Per}_1(e^{2\pi i p/q})$ should be understood as the mating of $M \setminus L_{-p/q}$ with $K_\omega \setminus S_p$, the filled–in Julia set $K_\omega$ of the fat $p/q$–rabbit, without the critical value sector $S_p$.

We present here a model for $\hat{\mathcal{R}}_\omega \subset \text{Per}_1(\omega)$ (inspired by Douady & Hubbard, Goldberg & Keen, Milnor, Tan Lei & Petersen) which is slightly different from what the conjecture suggests it should be.

In this talk: we shall focus on why everything in the sector $S_p$ has to be left out, or in other words why this sector corresponds to obstructed parameters.

Our model suggests that $\text{Per}_1(\omega)$ should be described as a “doctored” mating.
Consider again the polynomial $P_\omega$ and let $\phi_\omega$ be the Fatou coordinate normalized so that $\phi_\omega \circ P_\omega = 1/q + \phi_\omega$ and $\phi_\omega(-\omega/2) = 0$.

Let $\bigcup_{j=0}^{q-1} P_{1/q}^j$ be the $q$ c.c.s of $\phi_\omega^{-1}(\{z = x + iy : x > 1/q\})$ with 0 on their boundaries.

**Definition:** Two points $z_1 \in \partial P^j$ and $z_2 \in \partial P^k$ are called equivalent modulo $p/q$, written $z_1 \sim_{p/q} z_2$, if the following two conditions are satisfied:

1. $j + k = 2p \mod q$, and
2. $\phi_\omega(z_1) + \phi_\omega(z_2) = 2/q$. **

**
Definition: Let $\tilde{K}_\omega = \left( \hat{K}_\omega \setminus (S_p \cup \bigcup_{j=0}^{q-1} \mathcal{P}_j^{1/q}) \right) / \sim_{p/q}$

Let $\nu_\omega \subset \tilde{K}_\omega$ denote the scar after this operation. From the Fatou coordinate we then define a *bubble–tree*, denoted $\tilde{T}_\omega$, in $\tilde{K}_\omega$:

$$
\tilde{T}_\omega = \left( \phi_\omega^{-1}(\mathbb{R}) \cup \nu_\omega \cup \bigcup_{n \geq 0} P_\omega^{-n}(0) \right) \cap \tilde{K}_\omega.
$$
A model for $\hat{R}^\omega$

**Theorem:** There exists a bijective map, $\chi^\omega : \hat{R}^\omega \rightarrow \tilde{K}_\omega$, encoding dynamics, which is conformal on the interior of the domain of definition, and so that $(\chi^\omega)^{-1}$ is continuous along the bubble–tree $\tilde{T}^\omega$.

Proof by picture...
Consider a map $g_\sigma \in \hat{R}_\omega$. It has a parabolic fixed point, a basin of attraction $\Lambda^\omega$ and a corresponding $q$–cycle of components, $B_0, \ldots, B_{q-1}$, in the immediate basin.
Let $\phi_\sigma : \Lambda^\omega \to \mathbb{C}$ be a Fatou coordinate normalized so that

$$\phi_\sigma^{(j+p) \mod q} \circ g_\sigma = \phi_\sigma^j + 1/q$$

Let $\mathcal{P}_r^j \subset B_j$ be an attracting petal, so that $\phi_\sigma|_{\mathcal{P}_r^j}$ is conformal onto the right half plane $\mathbb{H}_r = \{z = x + iy : x > r\}$.
Let $R$ be smallest so that each $\phi_\sigma : \mathcal{P}_R^j \to \mathbb{H}_R$ is conformal.
Then (at least) one petal $\mathcal{P}_R^j$ contains a critical point in its boundary, called the closest critical point $c_1$ (with value $v_1$).
The other critical point is then denoted $c_2$ (with value $v_2$).
Further normalize/name so that \( c_1 \in B_0 \) and \( \phi_\sigma(c_1) = 0 \).

Define a dynamical, conformal conjugacy

\[
\eta_{\sigma,\omega} = \phi_\omega^{-1} \circ \phi_\sigma : \bigcup_{j=0}^{q-1} P_{0,\sigma}^j \to \bigcup_{j=0}^{q-1} P_{0,\omega}^j
\]

between the closures of the attracting flowers. Furthermore, \( \eta_{\sigma,\omega} \) extends by iterated lifting to the dynamics, until the second critical value \( \nu_2 \) is in the domain of \( \eta_{\sigma,\omega} \).

Then, \( \chi_\omega : \hat{\mathcal{R}}_\omega \to \hat{\mathcal{K}}_\omega \) is (essentially) defined by \( \chi_\omega(\sigma) = \eta_{\sigma,\omega}(\nu_2) \).

(... well, for some parameters, f.ex. degenerate parabolic parameters there is more to the story, but that's another story)
Our result on obstructed parameters:

**Theorem:** For all \( g_\sigma \) in \( \hat{R}_\omega \), \( \eta_{\sigma,\omega}(v_2) \in \hat{K}_\omega \setminus S_p \).

**Proof:** Assume that there exists a \( g_\sigma \) in \( \hat{R}_\omega \), so that \( x = \eta_{\sigma,\omega}(v_2) \in \hat{K}_\omega \cap S_p \).

By the Riemann-Hurwitz formula, \( x \) is not in \( B_p \). Assume \( x \) is pre–fixed or pre–critical of generation \( n > 1 \). Then \( g_\sigma \) is conjugate to \( P_\omega \) up to this level.

A hypothetical example with generation 3... on the blackboard
Some notation...

- Let $I_p$ be the curve in $\phi_{-1}^{-1}(\mathbb{R})$ which connects the parabolic fixed point and the critical value $v_2$. Note that $v_1 \in I_p$.
- Let $\hat{I}_p$ be the truncation of $I_p$ connecting $v_1$ to $v_2$.
- Let $C$ be the pre–image of $\hat{I}_p$ under $g_\sigma$.
- The closed curve $C$ is mapped 2–to–1 onto $\hat{I}_p$. It intersects the boundary of $B_0$ at two places, namely at two pre–images of the parabolic fixed point, of same generation $k = 1 + qm$, for some $m$.
- Let $\nu$ be a curve, “following Fatou equipotential–curves” (with the least number of intersections with $\partial B_0$) connecting these two intersections through the parabolic fixed point.
Pull back $C'$ $q - 1$ times (each time choosing pre–image in $C_{ext}$), to a simple closed curve $l_1$ passing through $B_p$, containing the parabolic fixed point and surrounding $\hat{l}_p$.

Denote by $V_1$ the topological disk with $l_1$ as its boundary.

By iterated pull–back with $g^q_\sigma$, we obtain a sequence of simple closed curves $l_n$, homotopic with respect to the set of iterated forward images of the critical points, pairwise disjoint except for the parabolic fixed point, and so that $l_n \to l_{n-1}$ 1–to–1 by $g^q_\sigma$.

Denote by $V_n$ the “croissant” bounded by $l_{n-1} \cup l_n \cup \{z_0\}$, let $\Omega = \bigcup_{n \geq 2} V_n \subseteq C_{ext}$ and $Q = \Omega / g^q_\sigma$.

Then $Q \cong \mathbb{C}/\mathbb{Z} \Rightarrow$ a contradiction.
Figure: A forbidden fat rabbit.
A relatively hyperbolic Julia set

Figure: A basilica and the fat rabbit.
A parabolic bitransitive Julia set

Figure: A “bitransitive fat rabbit”.
A parabolic capture Julia set

Figure: A “capture fat rabbit”. 
Thank you for your attention and thanks to the organizers!